III YEAR - V SEMESTER
COURSE CODE: 7BMAE1A

ELECTIVE COURSE - I (A) – GRAPH THEORY

Unit – I

Graphs – Definition and examples – Degrees – Sub graphs – Isomorphism – Ramsey Numbers – Independent Sets and Coverings – Intersection graphs and Line graphs – Matrices – Operations on Graphs.

Unit – II

Dergee Sequences – Graphic sequences – Walks, Trials and Paths – Connectedness and Components – Blocks – Connectivey – Eulerian Graphs – Hamiltonian Graphs.

Unit – III

Trees – Characterisation of Trees – Centre of a Tree – Matchings–Matchings in Bipartite Graphs.

Unit – IV

Planer graphs and properties – Characterization of Planer graphs – Thickness, crossing and outer planarity – Chromatic number and ChromaticIndex – The Five colour theorem and four colour problem.

Unit – V

Chromatic polynomials – Definitions and Basic properties of Directed Graph – Paths and Connections – Digraphs and Matrices – Tournaments.

Text Book:

1. Invitation to Graph Theory by Dr. S.Arumugam & S.Ramachandran, Scitech Publications (India) Pvt. Ltd,2001 .

Unit I	Chapter 2
Unit II	Chapters 3, 4 & 5
Unit III	Chapters 6 & 7
Unit IV	Chapter 8, Chapter 9, sections 9.1 to 9.3
Unit V	Chapter 9 section 9.4; Chapter 10

Book for Reference:

1. Graph Theory with Applications to Engineering and Computer Science byNarasinghDeo, Prentice Hall of India, New Delhi.

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Syllabus

Course code : TBMAE1A

Elective course : I(A) - Graph theory

unit - I

Graphs - Definition & Examples Degrees - Sub graphs - Isomorphism - Ramsey numbers - Independent sets & coverings - Intersection graphs and line graphs - matrices - Operations on Graphs.

unit - II

Degree sequences - Graphic sequences - walks, Trials and paths - connectedness and Components - Blocks - Connectivity - Eulerian Graphs - Hamiltonian Graphs.

unit - III

Trees - characterisation of trees - Centre of a tree - matching - matching in Bipartite Graphs

unit - IV

planer graphs and properties - characterization of planar graphs - Thickness, crossing and outer planarity - chromatic number and chromatic Index - The five colour theorem & four colour problem.

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Chromatic polynomials - Definition and basic properties of Directed Graph-paths & Connections - Digraphs and matrices - Tournaments.

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unit - I chapter 2
unit - II chapter 3, 4, 5.
unit - III chapter 6, 7
unit - IV chapter 8, 9.1; 9.3
unit - V chapter 9, 9.4, 10

Definition :

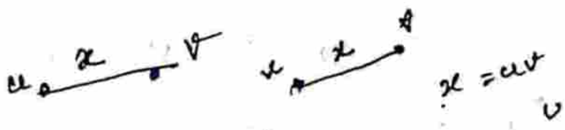
1

1. Graph :

A Graph G consists of a pair $(V(G), X(G))$ where $V(G)$ is a non-empty finite set whose elements are called points or vertices and $X(G)$ is a set of unordered pair of distinct elements of $V(G)$.

2. lines or edges :

The elements of $X(G)$ are called lines or edges of the graph.

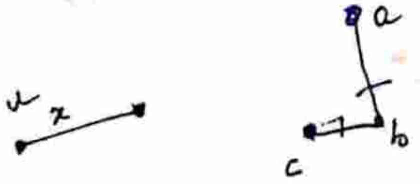


3. Adjacent :

If $x = \{u, v\} \in X(G)$, the line x is said to join u and v. we write $x = uv$ and we say that the points u and v are adjacent.

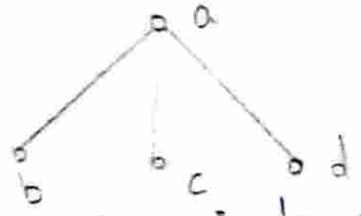
4. Adjacent lines :

We also say that the point u and the line x are incident with each other. If two distinct lines x and y are incident with a common point then they are called adjacent lines.



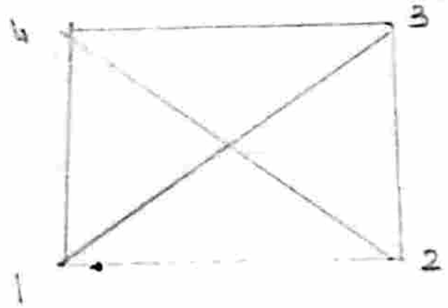
Examples.

1 Let $V = \{a, b, c, d\}$ and $X = \{\{a,b\}, \{a,c\}, \{a,d\}\}$ $G = (V, X)$ is a $(4, 3)$. This graph can be represented by the diagram.



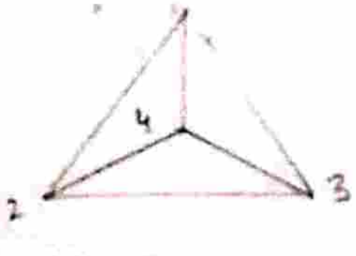
In this graph points a and b are adjacent whereas b and c are non-adjacent.

2. Let $V = \{1, 2, 3, 4\}$ and $X = \{\{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}\}$ $G = (V, X)$ is a $(4, 6)$ graph.



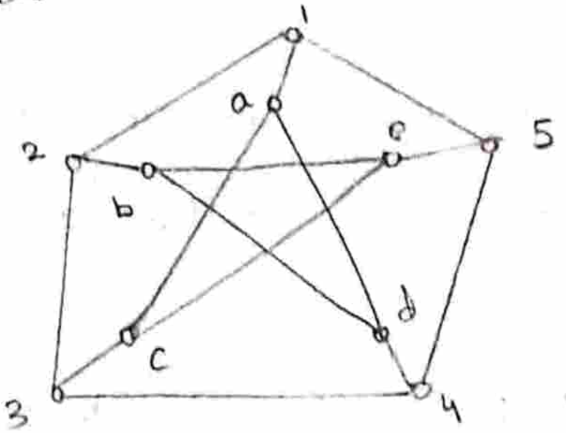
This graph is represented by the diagram given. Although the lines $\{1,2\}$ and $\{2,4\}$ intersect in the diagram, their intersection is not a point of the graph.

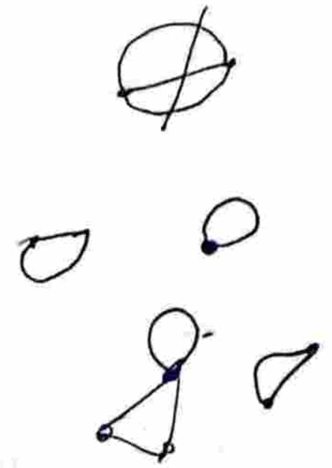
3. The $(10, 15)$ graph is called the Petersen graph.



Remark:

The definition of a graph does not allow more than one line joining two points. It also does not allow any line joining a point to itself. Such a line joining a point to itself is called a loop.





Definition:

If more than one line joining two vertices are allowed, the resulting object is called a <u>multigraph</u>. Lines joining the same points are called multiple lines. If further loops are also allowed, the resulting object is called a pseudo graph.

Example:

fig. 1 is a multigraph and fig 2 is a pseudo graph and fig 1.2 of the Konigsberg bridge problem is a multigraph.

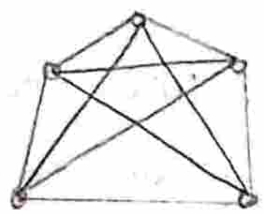
Remark:

Let G be a (p,q) graph. Then $q \le \binom{p}{2}$ and $q = \binom{p}{2}$ iff any two distinct points are adjacent.

Definition:

A graph in which any two distinct points are adjacent is called a complete graph.

The complete graph with p points is denoted by K_p.



K_3 is called a triangle. The graph is given. K_4 and K_5 are shown.

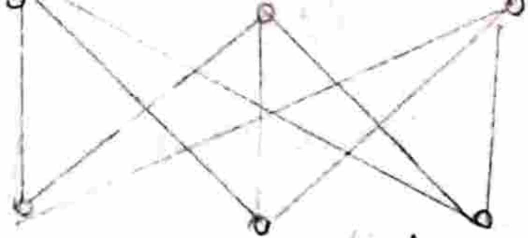
Definition:

A graph whose edge set is empty is called a null graph or a totally disconnected graph.

Definition:

A graph G is called labelled if its p points are distinguished from one another

(5)



The graph given are labelled graphs and the graph is unlabelled graph.

Definition.

A graph G is called a bigraph or bipartite graph if V can be partitioned into two disjoint subsets V_1 and V_2 such that every line of G joints a point of V_1 to a point of V_2. (V_1, V_2) is called a bipartition of G. If further G contains every line joining the points of V_1 to the points of V_2 then G is called a complete bigraph. If V_1 contains m points and V_2 contain n point then the complete bigraph G is denoted by $K_{m,n}$. The graph given in $K_{1,3}$. The graph is $K_{3,3}$. $K_{1,m}$ is called a star for $m \geq 1$

Degrees:

The degree of a point v_i in a graph G is the number of lines incident with v_i. The degree of v_i is denoted by $d_G(v_i)$ or deg v_i or simply $d(v_i)$.

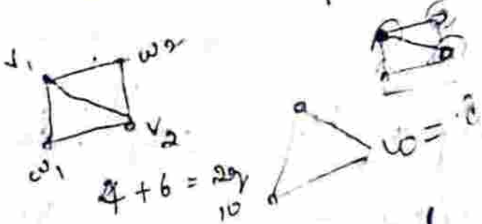
A point v of degree 0 is called a isolated point. A point v of degree 1 is called an end point.

Theorem 2.1

The sum of the degrees of the points of a graph G is twice the number of lines. $\sum_i \deg v_i = 2q$

proof:

Every line of G is incident with two points. Hence every line contributes 2 to the sum of the degrees of the points



Hence $\sum_i \deg v_i = 2q$.

Corollary;

In any graph G, the number of points of odd degree is even.

Let $v_1, v_2 \ldots v_k$ denote the points of odd degree $w_1, w_2 \ldots w_m$ denote the points of even degree in G.

$\sum_{i=1}^{k} \deg v_i + \sum_{i=1}^{m} \deg w_i = 2q$. which is even. further $\sum_{i=1}^{m} \deg w_i$ is even

Hence $\sum_{i=1}^{k} \deg v_i$ is also even

But $\deg v_i$ is odd for each i

Hence k must be even.

Definition for any graph G, we define

$\delta(G) = \min \{\deg v / v \in V(G)\}$ and

$\Delta(G) = \max \{\deg v / v \in V(G)\}$

If all the points of G have the same degree r then $\delta(G) = \Delta(G) = r$ and in this case G is called regular graph of degree r

For example the complete graph K_p is regular of degree $p-1$

Theorem 2.2

Every cubic graph has an even number of points.

Proof: Let G be a cubic graph with p points.

Then $\sum \deg v = 3p$ which is even by Thm 2.1

Hence p is even

Solved problems:

Problem 1

Let G be a $\{p,q\}$ graph all of whose points have degree k or $k+1$. If G has $t > 0$ points of degree k. Show that $t = p(k+1)-2q$.

Solution

Since G has t points of degree k, the remaining $p-t$ points have degree $k+1$. Hence

$$\sum_{v\in V} d(v) = tk + (p-t)(k+1)$$

$$tk + (p-t)(k+1) = 2q$$

$$tk + (pk + p - tk - t) = 2q$$

$$p(k+1) - t = 2q$$

$$t = p(k+1) - 2q$$

Problem 2 Show that in any group of two or more people, there are always two with exactly the same number of friends ...

Solution: (8)

We construct a graph G by taking the group of people as the set of points and joining two of them if they are friends. Then $\deg v$ = number of friends of v and hence we need only to prove that at least two points of G have the same degree.

Let $V(G) = \{v_1, v_2 \ldots v_p\}$

Clearly $0 \leq \deg v_i \leq p-1$ for each i

Suppose no two points of G have the same degree. Then the degrees of $v_1, v_2 \ldots v_p$ are the integers $0, 1, 2 \ldots, p-1$ in some order. However a point of degree $p-1$ is joined to every other point of G and hence no point can have degree zero which is contradiction.

Hence there exist two points of G with equal degree.

problem 3. prove that $\delta \leq \frac{2q}{p} \leq \Delta$

Solution.

Let $V(G) = \{v_1, v_2 \ldots v_p\}$

we have $\delta \leq \deg v_i \leq \Delta$ for all i

Hence $p\delta \leq \sum_{i=1}^{p} \deg v_i \leq p\Delta$

$$p\delta \leq 2q \leq p\Delta$$

$$\delta \leq \frac{2q}{p} \leq \Delta$$

problem 4

Let G be a k-regular bigraph with bipartition $\{V_1, V_2\}$ and $k>0$. prove that $|V_1| = |V_2|$

Solution: Since every line of G has one end in V_1 and the other end in V_2 it follows that

$$\sum_{v \in V_1} d(v) = \sum_{v \in V_2} d(v) = q$$

$$V = V_1 \cup V_2.$$

Also, $d(v) = k$ for all $v \in V$

Hence $\sum_{v \in V_1} d(v) = k|V_1|$ and $\sum_{v \in V_2} d(v) = k|V_2|$

So that $k|V_1| = k|V_2|$

Since $k>0$, we have $|V_1| = |V_2|$.

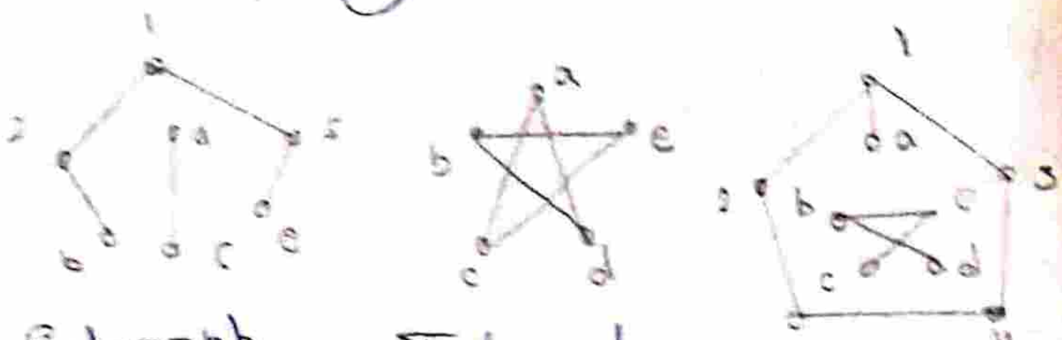
Subgraphs:

Definition:

A graph $H = (V_1, X_1)$ is called a subgraph of $G = (V, X)$ if $V_1 \subseteq V$ and $X_1 \subseteq X$. If H is a subgraph of G we say that G is a supergraph of H. H is called a spanning subgraph of G If $V_1 = V$. H is called an induced subgraph of G. If $X_2 \subseteq X$, then the subgraph of G with line set X_2 and having no isolated points is called the subgraph line induced (edge induced) by X_2 and is denoted by $G[X_2]$

Example:

Consider the peterson graph G.



Subgraph Induced Subgraph Spanning Subgraph

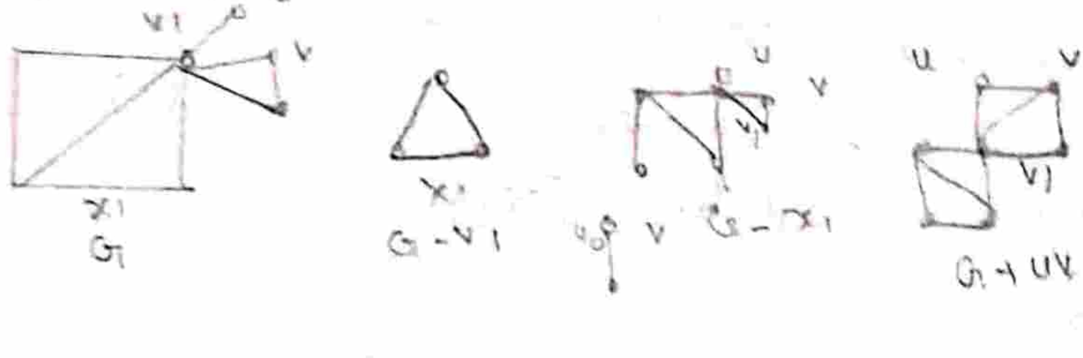
Definition:

Let $G = (V, X)$ be a graph. Let $v_i \in V$. The Subgraph of G obtained by removing the point v_i and all the lines incident with v_i is called the Subgraph obtained by the removal of the point v_i and is denoted by $G - v_i$

Thus if $G - v_i = (V_i, X_i)$ then $V_i = V - \{v_i\}$ and $X_i = \{x \mid x \in X$ and x is not incident with $v_i\}$

Definition.

Let $G = (V, X)$ be a graph. Let v_i, v_j be two points which are not adjacent in G. Then $G + v_i v_j = (V, X \cup \{v_i, v_j\})$ is called the graph obtained by the addition of the line $v_i v_j$ to G



Theorem 2.3

The maximum number of lines among all p point graphs with no triangles is $\left[\frac{p^2}{4}\right]$ ($[x]$ denotes the greatest integer not exceeding the real number x).

proof: The result can be easily verified for $p \le 4$ for $p > 4$, we will prove by induction separately for odd p and for even p.

part 1. for odd p

Suppose the result is true for all odd $p \le 2n+1$ Now let G be a (p, q) graph with $p = 2n+3$ and no triangles. If $q = 0$ then $q \le \left[\frac{p^2}{4}\right]$. Hence let $q > 0$. Let u and v be a pair of adjacent points in G. The subgraph $G' = G - \{u, v\}$ has $2n+1$ points and no triangles. Hence by induction hypothesis,

$$q(G') \le \left[\frac{(2n+1)^2}{4}\right] = \left[\frac{4n^2+4n+1}{4}\right]$$

$$= \left[n^2+n+1/4\right] = n^2+n \quad \ldots (1)$$

Since G has no triangles, no point of G' can be adjacent to both u and v in G ... (2)

Now, lines in G are three types.

i) lines of G' ($\le n^2+n$ in number by (1))

ii) lines between G' and $\{u, v\}$ ($\le 2n+1$ in number by (2))

iii) line uv (12)

Hence $q \le (n^2+n) + (2n+1) + 1 = n^2+3n+2$

$= \frac{1}{4}(4n^2+12n+8)$

$= \left(\frac{4n^2+12n+9}{4} - \frac{1}{4}\right)$

$= \left[\frac{(2n+3)^2}{4}\right] = \left[\frac{p^2}{4}\right]$

Also for $p = 2n+3$, the graph $K_{n+1,n+2}$ has no triangles and has $(n+1)(n+2) = n^2+3n+2 = \left(\frac{p^2}{4}\right)$ lines. Hence this maximum q is attained.

Case 2. for even p

Suppose the result is true for all even $p \le 2n$

Now let G be a (p,q) graph with $p = 2n+2$ and no triangles. As before, let u and v be a pair of adjacent points in G and let $G' = G - \{u,v\}$

Now let G be a (p,q) graph with $p = 2n+2$ and no triangles. As before, let u and v be a pair of adjacent points in G and let $G' = G - \{u,v\}$

Now G' has $2n$ points and no triangles.

Hence by hypothesis

$$q(G') \le \left[\frac{(2n)^2}{4}\right] = n^2 \quad \cdots (3)$$

lines in G are of three types.

i) lines of G' ($\le n^2$ in number by (3))

ii) lines between G' and $\{u,v\}$ ($\le 2n$ in number by an argument similar to (2))

iii) Let no uv (13)

Hence $q \le n^2+2n+1 = (n+1)^2 = \frac{(2n+2)^2}{4} = \left[\frac{p^2}{4}\right]$

Hence the result holds for even p also

We see that for $p = 2n+2$, $K_{n+1, n+1}$ is a $(p, [p^2/4])$ graph without triangles.

Isomorphism:

Definition:

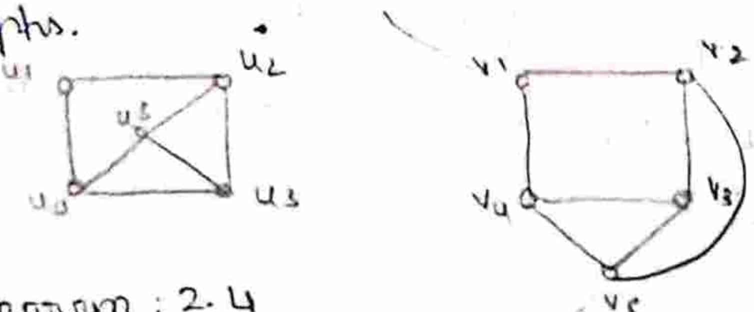
Two graphs $G_1 = (V_1, X_1)$ and $G_2 = (V_2, X_2)$ are said to be isomorphic if there exists a bijection $f: V_1 \to V_2$ such that u, v are adjacent in G_1 iff $f(u), f(v)$ are adjacent in G_2.

If G_1 is isomorphic to G_2 we write $G_1 \cong G_2$

The map f is called an isomorphism from G_1 to G_2

Examples:

The two graphs given are isomorphic. $f(u_i) = v_i$ is an isomorphism between these two graphs.



Theorem : 2.4

Let f be an isomorphism of the graph $G_1 = (V_1, X_1)$ to the graph $G_2 = (V_2, X_2)$ Let $v \in V_1$. Then $\deg v = \deg f(v)$.

Solution:

A point $u \in V_1$ is adjacent to v in G_1 iff $f(u)$ is adjacent to $f(v)$ in G_2. Also

f is a bijection. Hence the number of points in V_1 which are adjacent to v is equal to the number of points in V_2 which are adjacent to $f(v)$. Hence $\deg v = \deg f(v)$

Definition:

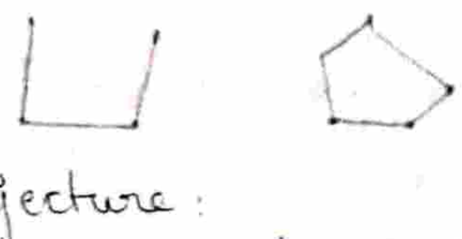
An isomorphism of a graph G onto itself is called an automorphism of G.

Hence $\Gamma(G)$ is a group and is called the automorphism group of G.

27/6/19

Self complementary graph:

Let $G = (V, X)$ be a graph. The complement of $\bar{G}$ of G is defined to be the graph which has V as its set of points and two points are adjacent in $\bar{G}$ iff they are not adjacent in G. G is said to be a self complementary graph if G is isomorphic to $\bar{G}$.



Ulam's conjecture:

Let G and H be two graphs with p points $(p > 2)$ say $v_1, v_2 \ldots v_p$ and $w_1, w_2 \ldots w_p$. The subgraph $G_i = G - v_i$ and $H_i = H - w_i$ are isomorphic the graph G and H are isomorphic.

Ulam's conjecture is also known as reconstruction conjecture.

problem : 1

prove that any self complementary graphs has $4n$ or $4n+1$ points.

Solution :

Let $G = (V(G), X(G))$ be a self complementary graph with p points

G is self complementary, $G \cong \bar{G}$

$$|X(G)| = |X(\bar{G})|$$

$$|X(G)| + |X(\bar{G})| = \binom{p}{2} = \frac{p(p-1)}{2}$$

$$2|X(G)| = \frac{p(p-1)}{2}$$

$$|X(G)| = \frac{p(p-1)}{4}$$ is an integer further one of p or $p-1$ is odd.

p or $p-1$ is a multiple of 4

p is of the form $4n$ or $4n+1$.

problem : 2 prove that $\Gamma(G) = \Gamma(\bar{G})$

Solution :

Let $f \in \Gamma(G)$ and Let $u, v \in V(G)$

Then u, v are adjacent in $\bar{G} \Leftrightarrow u, v$ are not adjacent in G

$\Leftrightarrow f(u), f(v)$ are not adjacent in G

(Since f is an automorphism of G)

$\Leftrightarrow f(u), f(v)$ are adjacent in $\bar{G}$

f is an automorphism of G

$f \in \Gamma(\bar{G})$ and hence $\Gamma(G) \subseteq \Gamma(\bar{G})$

$\Gamma(\bar{G}) \subseteq \Gamma(G)$

$$\Gamma(G) = \Gamma(\bar{G})$$

Ramsey numbers :

G contains three mutually adjacent points or three mutually non-adjacent points. Equivalently G or $\bar{G}$ contains a triangle.

Theorem : 2.5

for any graph G with 6 points, G or $\bar{G}$ contains a triangle.

Proof:

Let v be a point of G.

Since G contains 5 points other than v. v must be either adjacent to three points in G or non-adjacent to three points in G.

v must be adjacent to three points either in G or $\bar{G}$.

Let us assume that v is adjacent to three points u_1, u_2, u_3 in G.

If two of these three points are adjacent points form a triangle in $\overline{G}$

Hence G or $\overline{G}$ Contains a triangle.

Thus 6 is the Smallest positive integer Such that any graph G on 6 points Contains K_3 or $\overline{K_3}$.

$r(3,3) = 6$. The numbers $r(m,n)$ are called Ramsey numbers.

Solved problems:

problem 1: prove that $r(m,n) = r(n,m)$

Solution.

$$\text{Let } [r(m,n)] = s$$

G and $\overline{G}$ contain s points

$$\text{Since } r(m,n) = s$$

$\overline{G}$ has either K_m or $\overline{K_n}$ as an induced Subgraph

G has K_n or $\overline{K_m}$ as an induced Subgraph.

Thus an arbitrary graph on s points contain K_n or $\overline{K_m}$ as an induced Subgraph

$$r(n,m) \leq s$$

$$r(n,m) \leq r(m,n)$$

Interchanging m and n $r(m,n) \leq r(n,m)$

$$r(m,n) = r(n,m)$$

Problem: 2

prove that $r(2,2)=2$

Solution:

Let G be a graph on 2 points

$V(G) = \{u, v\}$

Then u and v are either adjacent in G or adjacent in $\overline{G}$

G or $\overline{G}$ contains K_2

If G is any graph on two points, then G or $\overline{G}$ contains K_2

2 is the least positive integer with this property

$$r(2,2) = 2.$$

Independent sets and coverings:

Definition.

A covering of a graph $G=(V,X)$ is a subset K of V such that every line of G is incident with a vertex in K.

A covering K is called a minimum covering if G has no covering K' with $|K'| < |K|$.

The number of vertices in a minimum covering of G is called the

Covering number of G and is denoted by β.

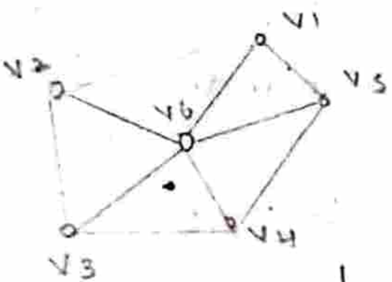
A subset S of V is called an independent set of G if no two vertices of S are adjacent in G.

An independent set S is said to be maximum if G has no independent set S' with $|S'| > |S|$.

The number of vertices in a maximum independent set is called the independence number of G and is denoted by α.

I.d -> no two vertice ~~non~~ adi
MId ->
c -> line cover.
min c -> max. line. cover.

Example:



$\{v_6\}$ is an independent set.
$\{v_1, v_3\}$ is a maximum independent set.
$\{v_1, v_2, v_3, v_4, v_5\}$ is a covering
$\{v_2, v_6, v_4, v_5\}$ is a minimum covering.

Theorem: 2.6

A set $S \subseteq V$ is an independent set of G if and only if $V-S$ is a covering of G

A subset S of V is called an independent set of G iff no two vertices of S are adjacent in G.

iff every line of S is incident with atleast one point of $V-S$

iff $V-S$ is a covering of G.

Corollary:

$$\alpha + \beta = p$$

proof: Let S be a maximum independent set of G and K be a minimum covering of G.

$$|S| = \alpha \text{ and } |K| = \beta$$

$V-S$ is a covering of G and K is a minimum covering of G

$$|K| \leq |V-S|$$

$$\beta \leq p - \alpha$$

$$\beta + \alpha = p \quad \dots (1)$$

$V-K$ is an independent set and S is a maximum independent set

$$\alpha \geq p - \beta$$

$$\alpha + \beta \geq p \quad \ldots \text{(2)}$$

From (1) and (2)

$$\alpha + \beta = p$$

Definition :

Line covering :

A line covering of G is a subset L of X such that every vertex is incident with a line of L.

Line covering number :

The number of lines in a minimum line covering of G is called the line covering number of G and is denoted by β'

Independent :

A set of lines is called independent if no two of them are adjacent.

Edge independence number :

The number of lines in a maximum independent set of lines is called the edge independence number and is denoted by α'

Result :

$$\alpha' + \beta' = p$$

proof :

Let S be a maximum independent set of lines of G

$$|S| = \alpha'$$

Let M be set of lines, one incident lines for each of the $p - 2\alpha'$ points of G not covered by any line of S.

$S \cup M$ is a line covering of G.

$$|S \cup M| \geq \beta'$$

$$\alpha' + p - 2\alpha' \geq \beta'$$

$$p \geq \alpha' + \beta' \quad \cdots (1)$$

T be a minimum line cover of G

$$|T| = \beta'$$

T cannot have a line x both of whose ends are also incident with lines of T other than x

$T - \{x\}$ will become a line covering of G.

$G[T]$, the spanning subgraph of G induced by T, is the union of stars.

Each line of T is incident with at least one end point of $G[T]$.

Let W be a set of end points of $G[T]$ consisting of exactly one end point for each line of T

$|W| = |T| = \beta'$. Each star has exactly one point not in W

$p = |W| +$ (number of stars in $G[T]$)

$p = \beta' +$ (number of stars in $G[T]$) .. (2)

By choosing one line from each star of $G[T]$, we get a set of independent lines of G

$\alpha' \geq$ (number of stars in $G[T]$)

(2) $\Rightarrow$ $p \leq \beta' + \alpha'$

(1) $\Rightarrow$ $\alpha' + \beta' = p$

This completes the proof.

Intersection Graphs and line graphs.

Definition:

Intersection graph:

Let $F = \{S_1, S_2 \ldots S_p\}$ be a non-empty family of distinct non empty subsets of a given set S. The intersection graph of F. denoted by $\Omega(F)$.

The set of points of V of $\Omega(F)$ is F itself and two points S_i, S_j are adjacent if $i \neq j$ and $S_i \cap S_j \neq \phi$.

A graph G is called an intersection graph on S if there exists a family F of subsets of S such that G is isomorphic to $\Omega(F)$.

Theorem : 2.7

Every graph is an intersection graph.

proof:

Let $G = (V, X)$ be a graph

Let $V = \{v_1, v_2 \ldots v_p\}$. Let $S = V \cup X$

for each $v_i \in V$, Let $S_i = \{v_i\} \cup \{x \in X / v_i \in x\}$.

Clearly, $F = \{S_1, S_2 \ldots S_p\}$ is a family of distinct non-Empty subset of S.

If v_i, v_j are adjacent in V.

$v_i v_j \in S_i \cap S_j$

$S_i \cap S_j \neq \phi$

Common to $S_i \cap S_j$ is the line joining v_i and v_j so that v_i, v_j are adjacent in G

$f : V \to F$ defined by $f(v_i) = S_i$

is an isomorphism of G to $\Omega(F)$

G is an intersection graph.

Definition :

Line graph:

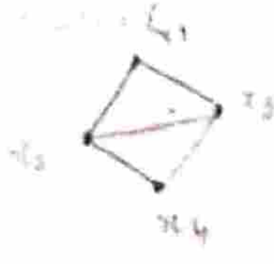
Let $G = (V, X)$ be a graph with $X \neq \phi$

Then X can be as a family of 2 element subsets of V.

The intersection graph $\Omega(X)$ is called the line graph of G and is denoted by $L(G)$.

The points of $L(G)$ are the lines of G and two points in $L(G)$ are adjacent iff the corresponding lines are adjacent in G

Example :



Theorem : 8.8

Let G be a (p, q) graph. Then $L(G)$ is a (q, q_L) graph where $q_L = \frac{1}{2}\left(\sum_{i=1}^{p} d_i^2\right) - q$

Proof:

By definition, number of points in $L(G)$ is q

To find the number of lines in $L(G)$

Any two of the d_i lines incident with v_i are adjacent in $L(G)$

$\frac{d_i(d_i-1)}{2}$ lines of $L(G)$

$$q_L = \sum_{i=1}^{p} \frac{d_i(d_i-1)}{2}$$

$$= \frac{1}{2}\left(\sum_{i=1}^{p} d_i^2\right) - \frac{1}{2}\left(\sum_{i=1}^{p} d_i\right)$$

$$= \frac{1}{2}\left(\sum_{i=1}^{p} d_i^2\right) - \frac{1}{2}(2q)$$

$$q_L = \frac{1}{2}\left(\sum_{i=1}^{p} d_i^2\right) - q$$

Theorem : 8.9 Whitney :

Let G and G' be connected graphs with isomorphic line graphs.

Then G and G' are isomorphic unless one is K_3 and other is $K_{1,3}$.

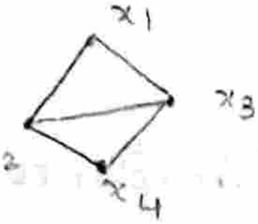
Definition :

Line graph :

A graph G is called a line graph if $G \cong L(H)$ for some graph H.

Example :

$K_4 - x$ is a line graph.



Theorem : 2.10 (Beineke)

G is a line graph iff none of the nine graphs.

is an induced subgraph of G.

Matrices :

Definition :

Adjacency matrix :

Let $G = (V, X)$ be a (p, q) graph. Let $V = \{v_1, v_2 \dots v_p\}$. Then $p \times p$ matrix

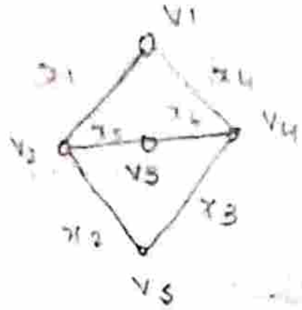
$$A = (a_{ij})$$

$$a_{ij} = \begin{cases} 1 & \text{if } v_i, v_j \text{ are adjacent} \\ 0 & \text{otherwise} \end{cases}$$

is called the adjacency matrix of graph

Example:

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$



Incidence matrix:

let $G = (V, X)$ be a (p, q) graph

Let $V = \{v_1, v_2 \dots v_p\}$ $X = \{x_1, x_2 \dots x_q\}$

$p \times q$ matrix $B = (b_{ij})$

$$b_{ij} = \begin{cases} 1 & \text{if } v_i \text{ is incident with } x_j \\ 0 & \text{otherwise} \end{cases}$$

is called the incidence matrix of the graph.

Example:

$$B = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

Operations on Graphs:

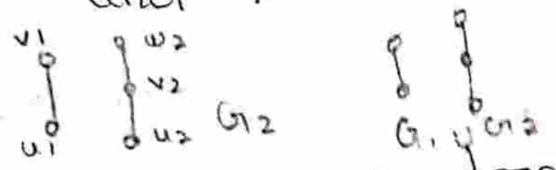
Definition:

let $G_1 = (V_1, X_1)$ and $G_2 = (V_2, X_2)$ be two graphs with $V_1 \cap V_2 = \phi$

i) union:

The union of $G_1 \cup G_2$ to be (V, X)

where $V = V_1 \cup V_2$ and $X = X_1 \cup X_2$.

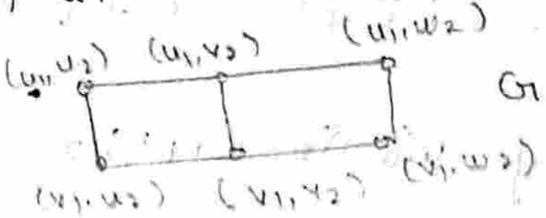


ii) Sum:

The sum $G_1 + G_2$ as $G_1 \cup G_2$ together with all the line joining points of V_1 to points of V_2.

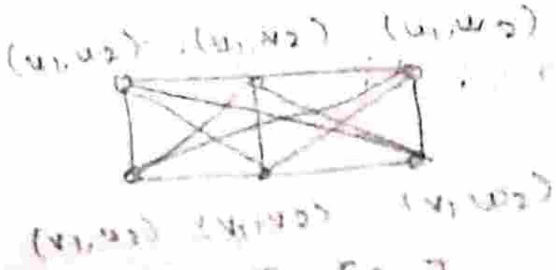
iii) product:

The product $G_1 \times G_2$ as having $V = V_1 \times V_2$ and $u = (u_1, u_2)$ and $v = (v_1, v_2)$ are adjacent if $u_1 = v_1$ and u_2 is adjacent to v_2 in G_2 or u_1 is adjacent to v_1 in G_1 and $u_2 = v_2$



iv) Composition:

The composition $G_1[G_2]$ as having $V = V_1 \times V_2$ and $u = (u_1, u_2)$ and $v = (v_1, v_2)$ are adjacent if u_1 is adjacent to v_1 in G_1, or ($u_1 = v_1$ and u_2 is adjacent to v_2 in G_2.



$G_1[G_2]$

Theorem: 2.11

Let G_1 be a (p_1, q_1) graph and G_2 a (p_2, q_2) graph.

i) $G_1 \cup G_2$ is a (p_1+p_2, q_1+q_2) graph

ii) $G_1 + G_2$ is a $(p_1+p_2, q_1+q_2+p_1p_2)$ graph

iii) $G_1 \times G_2$ is a $(p_1p_2, q_1p_2+q_2p_1)$ graph

iv) $G_1[G_2]$ is a $(p_1p_2, p_1q_2+p_2^2q_1)$ graph.

Proof:

i) Let $G_1 = (p_1, q_1)$ and $G_2 = (p_2\ q_2)$

The union $G_1 \cup G_2$ to be (p, q) where

$p = p_1 \cup p_2$ and $q = q_1 \cup q_2$

$G_1 \cup G_2$ is a (p_1+p_2, q_1+q_2) graph.

ii) number of lines in G_1+G_2

= number of lines in G_1 + number of lines in G_2 + number of lines joining points of V_1 to points of V_2,

$= q_1 + q_2 + p_1p_2$

$G_1 + G_2$ is a $(p_1+p_2, q_1+q_2+p_1p_2)$ graph

$p_1 p_2$

Now, let $(u_1, u_2) \in V_1 \times V_2$

The points adjacent to (u_1, u_2) are (u_1, v_2) where u_2 is adjacent to v_2 and (v_1, u_2) where v_1 is adjacent to u_1

$$\deg(u_1, u_2) = \deg u_1 + \deg u_2$$

The total no of lines in $G_1 \times G_2$

$$= \frac{1}{2}\left[\sum_{i,j} \deg(u_i) + \deg(v_j)\right]$$

$$= \frac{1}{2} \sum_{i=1}^{p_1} \sum_{j=1}^{p_2} (\deg u_i + \deg v_j)$$

where $u_i \in V_1$, $v_j \in V_2$

$$= \frac{1}{2} \sum_{i=1}^{p_1} \left(p_2 \deg u_i + \sum_{j=1}^{p_2} \deg v_j\right)$$

$$= \frac{1}{2} \sum_{i=1}^{p_1} (p_2 \deg u_i + 2q_2)$$

$$= \frac{1}{2}(p_2 2q_1 + p_1 2q_2)$$

$$= p_2 q_1 + p_1 q_2$$

iv)

Unit - 2

Degree Sequence:

partition :

A partition of a non-negative integer n is a finite set of non negative integers $d_1, d_2 \ldots d_p$ whose sum is n.

We denote this partition by $(d_1, d_2, \ldots d_p)$

partition on degree sequences:

Let G be a (p, q) graph. The partition of $2q$ as the sum of the degrees of its points is called the partition on the degree sequence of the graph G

Graphical partition or graphic sequence :

A partition $p = (d_1, d_2 \ldots d_p)$ of n into p parts is said to be a graphical partition or graphic sequence if there exists a graph G whose points have degree d_i and G is called realisation of p.

Example :

The partition $p = (2,1,1)$ of 4 is graphical. $K_{1,2}$ is the unique realisation of p.

Solved problem :

problem : 1

Show that the partition $p = (7,6,5,4,3,2)$ is not graphic.

Solution.

Suppose p is graphic

Let G be a realiation of p

G has six points.

Hence, the maximum degree of any point in G is 5 which is contradiction

p is not graphic.

problem : 2 Show that the partition $p = (6,6,5,4,3,3,1)$ is not graphic.

Solution

Suppose p is graphic

let G be a realisation of p

G has seven points

Since two points of G have degree 6

Each of these two points is adjacent

with every other point of G

The degree of each vertex in G is

at least 2 so that G has no point of

degree 1 which is contradiction

p is not graphic.

Graphic Sequence :

Theorem : 3.1

A partition $p = (d_1, d_2 \dots d_p)$ of

an even number into p parts with

$p-1 \geq d_1 \geq d_2 \geq \dots \geq d_p$ is graphical iff the

modified partition $p' = (d_2 - 1, d_3 - 1 \dots d_{d_1+1} - 1,$

$d_{d_1+2} \dots d_p)$ is graphical.

Proof

Suppose p' is a graphic sequence

let G' be a graph with vertex set

$\{v_2, v_3 \dots v_p\}$ such that

$d(v_2) = d_2 - 1 \dots d(v_p) = d_p$.

Let G be the graph obtained by G' adding a new vertex v_1 and making it adjacent to $v_2, v_3 \dots v_{d_1+1}$.

Clearly the partition of G is p and hence p is a graphic sequence.

Conversely, suppose p is graphical.

Let $G = (V, X)$ be a realisation of P

Let $V = \{v_1, v_2 \dots v_p\}$ with $\deg v_i = d_i$. If v_1 is adjacent to $v_2, v_3 \dots v_{d_1+1}$ Then $G - v_1$ is a realisation of P'.

If the graph G does not have this property we will show that from G. We can construct another realisation of P having this property.

Hence assume that in G. v_1 is not adjacent to all vertices $v_2, v_3 \dots v_{d_1+1}$.

Then there exist two vertices v_i and v_j such that $d_i > d_j$. v_1 is adjacent with v_j but not adjacent with v_i.

Since $d_i > d_j$ there exist a vertex v_k such that v_k is adjacent with v_i but not adjacent with v_j

Let G' be the graph obtained from G by deleting the lines $v_1 v_j$, v_i, v_k and adding the lines $v_1 v_i$ and $v_j v_k$.

Clearly, G' is also realisation of P in which v_1 is adjacent with v_i but not with v_j.

By repeating the process we obtain a realisation of P in which v_1 is adjacent to $v_2, v_3 \ldots v_{d_1+1}$

Hence the theorem is proved.

Remarks:

1. Any two isomorphic graphs determine the same partition.

The two non-isomorphic graphs.

Same partition $(3, 2, 1, 1, 1)$

2. If the partition $(d_1, d_2 .. d_p)$ of n is graphical. n is even $d_i \leq p-1$

For Example, the partition $(3, 3, 3, 1)$ of 10 is not graphical.

Examples:

1. Let $P = (6, 6, 5, 4, 3, 3, 1)$

$(5, 4, 3, 2, 2, 0) = P_1$

$P' = (5, 4, 3, 2, 2, 0)$

$P'' = (3, 2, 1, 1, -1)$ which involves a negative Summand.

p is not graphical.

2. Let $P = (4, 4, 4, 2, 2, 2)$

$P' = (3, 3, 1, 1, 2)$

$P_1 = (3, 3, 2, 1, 1)$

$P'' = (2, 1, 0, 1)$

$P_2 = (2, 1, 1, 0)$

P_2 is graphical and p is graphical.

Realisation of P_2, P_1 and P

Theorem: 3.2

If a partition $p = (d_1, d_2 .. d_p)$ is graphical with $d_1 \geq d_2 \geq \cdots \geq d_p$

then $\sum_{i=1}^{p} d_i$ is even and $\sum_{i=1}^{k} d_i \le k(k-1) + \sum_{i=k+1}^{p} \min\{k, d_i\}$ for $1 \le k \le p$.

proof:

let $G = (V, X)$ be realisation of P

$V = \{v_1, v_2, \ldots v_p\}$ and $\deg v_i = d_i$

$\sum_{i=1}^{p} d_i = 2q$ which is even.

The sum $\sum_{i=1}^{k} d_i$ is the sum of the degrees of the points $v_1, v_2, \ldots v_k$. It can divided into two parts

The first part being contribution to this sum by line joining the points $v_1, \ldots v_k$

The second part being contribution to this sum by lines joining one of the points $v_{k+1}, v_{k+2} \ldots v_p$ with points of $\{v_1, \ldots v_k\}$

The first part is $\le k(k-1)$

The second part is $\le \sum_{i=k+1}^{p} \min\{k, d_i\}$

$$\sum_{i=1}^{k} d_i \le k(k-1) + \sum_{i=k+1}^{p} \min\{k, d_i\}$$

connectedness :

Walks Trails and paths :

Walks :

A walk of a graph G is an alternating Sequence of points and lines $v_0, x_1, v_1, x_2, v_2 \ldots v_{n-1}, x_n, v_n$ beginning and ending with points such that each line x_i is incident with v_{i-1} and v_i.

The walk is $v_1, v_2, v_3, v_4, v_2, v_1, v_2, v_5$.

Terminal point :

The walk joins v_0 and v_n and it is called $v_0 - v_n$ walk v_n is called the terminal point.

Initial point :

The walk joins v_0 and v_n and it is called $v_0 - v_n$ walk. v_0 is called the initial point.

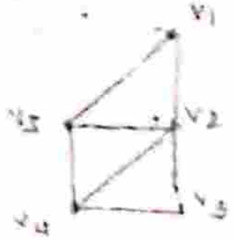
Length of the walk :

The walk is also denoted by $v_0, v_1 \ldots v_n$ the lines of the walk being self evident n: the number of lines in the walk is called length of this walk

A single point is considered as a walk of length 0.

Trail :

A walk is called a trail if all its lines are distinct.



$v_1, v_2, v_4, v_3, v_2, v_5$ is a trail but not a path.

path :

A walk is called a path if all its points are distinct.

v_1, v_2, v_4, v_5 is a path

Closed :

A $v_0 - v_n$ walk is called closed if $v_0 = v_n$.

Cycle :

A closed walk $v_0, v_1, v_2 \ldots v_n = v_0$ in which $n \geq 3$ and $v_0, v_1 \ldots v_{n-1}$ are distinct is called a cycle of length n

The graph consisting of a cycle of length n is denoted by C_n

Example:

C_3 is called a triangle.

Theorem: 4.1

In a graph G, any u-v walk contains a u-v path.

proof:

We prove the result by induction on the length of the walk.

Any walk of length 0 or 1 is obviously path. Now, assume the result for all walks of length less than n.

Let $u = u_0, u_1, \ldots u_n = v$ be a $u-v$ walk of length n.

If all the points of the walk are distinct is already a path.

If not, there exits i and j such that $0 \leq i < j \leq n$ and $u_i = u_j$

Now $u = u_0 \ldots u_i, u_{j+1} \ldots u_n = v$ is a $u-v$ walk of length less than n which contains a $u-v$ path by induction hypothesis.

Theorem: 4.2

If $\delta \geq k$, then G has a path of length k.

proof.

Let v_1 be an arbitrary point

choose v_2 adjacent to v_1

since $\delta \geq k$ there exits atleast

$k-1$ vertices other than v_1, which are

adjacent to v_2. choose $v_3 \neq v_1$

Such that v_3 is adjacent to v_2

In general having chosen $v_1 v_2 \ldots v_i$

where $i \leq \delta$ there exits a point

$v_{i+1} \neq v_1, v_2 \ldots v_i$ such that

v_{i+1} is adjacent to v_i.

This process yields a path of length

k in G.

Theorem. 4.3

A closed walk of odd length

Contains a cycle.

proof.

Let $v = v_0, v_1 \ldots v_n = v$ be a closed

walk of odd length.

Hence $n \geq 3$. If $n = 3$ this walk

is itself the cycle C_3 and hence the

theorem is trivial.

Now assume the result for all

walks of length less than n

If the given walk of length n is itself a is cycle there is nothing to prove.

If not there exist two positive Integers i and j such that $i < j$,

$\{i, j\} \neq \{0, n\}$ and $v_i = v_j$

Now $v_i, v_{i+1} \dots v_j$ and $v - v_0, v_1 \dots v_j, v_{j+1} - v_n = v$ are closed walks contained in the given walk and the sum of their lengths is n.

Since n is odd at least one of these walks is of odd length which by induction hypothesis contains a cycle.

Solved problem:

problem: 1

If A is the adjacency matrix of a graph with $V = \{v_1, v_2 \dots v_p\}$ prove that for any $n \geq 1$ the $(i, j)^{th}$ entry of A^n is the number of $v_i - v_j$ walks of length n in G.

Solution:

We prove the result by induction on n.

The number of $v_i - v_j$ walks of length 1

$$= \begin{cases} 1 & \text{if } v_i \text{ and } v_j \text{ are adjacent} \\ 0 & \text{otherwise} \end{cases}$$

$= a_{ij}$

Hence the result is true for $n = 1$

We now assume that the result is true for $n-1$

Let $A^{n-1} = (a_{ij}^{(n-1)})$ so that $a_{ij}^{(n-1)}$ is number of $v_i - v_j$ walks of length $n-1$ in G.

Now $A^{n-1} A = (a_{ij}^{(n-1)})(a_{ij})$

Hence $(i,j)^{th}$ entry of $A^n = \sum_{k=1}^{p} a_{ik}^{(n-1)} a_{kj}$ (1)

Also every $v_i - v_j$ walk of length n in G. Consists of a $v_i - v_k$ walk of length $n-1$ followed by a vertex v_j which is adjacent to v_k

Hence if v_j is adjacent to v_k then $a_{kj} = 1$ and $a_{ij}^{(n-1)} a_{kj}$ represents the number of $v_i - v_j$

walks of length n whose last edge is $v_k v_j$. Hence the right side of (1) gives the number of $v_i - v_j$ walks of length n in G. This complete the induction and the proof

Connectedness and Components:

Connected:

Two points u and v of a graph G are said to be connected if there exists a $u-v$ path in G.

A graph G is said to be connected if every pair of its points are connected.

Disconnected:

A graph which is not connected is said to be disconnected.

Example:

For $n > 1$, the graph $\overline{K_n}$ consisting of n points.

No union is disconnected. The union of two graphs is disconnected.

Components:

Let G_i denote the induced subgraph of G with vertex set V_i. Clearly the subgraphs $G_1, G_2 \ldots G_n$ are connected and are called the components of G.

Example:

Clearly a graph G is connected iff it has exactly one component.

A Example graph gives a disconnected graph with 5 components.

Theorem : 4.4

A graph G with p points and $\delta \geq \frac{p-1}{2}$ is connected.

Proof :

Suppose G is not connected.

Then G has more than one component

Consider any component $G_1 = (V_1, X_1)$ of G

let $v_1 \in V_1$ since $\delta \geq \frac{p-1}{2}$ there exist at least $\frac{p-1}{2}$ points in G_1 adjacent to v_1

Hence V_1 contains at least $\frac{p-1}{2} + 1 = \frac{p+1}{2}$ points

Thus each component of G contains at least $\frac{p+1}{2}$ points and G has at least two components.

Hence number of points in $G \geq p+1$

Which is contradiction.

Hence G is connected.

Theorem: 4.5

A graph G is connected iff for

any partition of V into subset V_1 and V_2 there is a line of G joining a point of V_1 to a point of V_2.

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proof:

Suppose G is connected

Let $V = V_1 \cup V_2$ be a partition of V into two subsets.

Let $u \in V_1$ and $v \in V_2$. Since G is connected, there exists a u-v path in G say.

$u = v_0, v_1, v_2 \ldots v_n = v$

Let i be the least positive integer such that $v_i \in V_2$ (Such an i exists since $v_n = v \in V_2$).

Then $v_{i-1} \in V_1$ and v_{i-1}, v_i are adjacent.

Thus there is a line joining $v_{i-1} \in V_1$ and $v_i \in V_2$.

To prove the converse, Suppose G is not Connected. Then G Contains at least two Components.

Let V_1 denote the Set of all Vertices of one Component and V_2 the remaining vertices of G.

clearly $V = V_1 \cup V_2$ is a partition of V and there is no line joining any point of V_1 to any point of V_2

Hence the theorem.

Theorem : 4.6

If G is not connected then $\overline{G}$ is connected.

proof :

Since G is not connected, G has more than one component.

Let u, v be any two points of G.

We will prove that there is a $u-v$ path in $\overline{G}$.

If u, v belong to different components in G, they are not adjacent in G and hence they are adjacent in $\overline{G}$.

If u, v lie in the same component of G, choose w in a different component.

Then u, w, v is a $u-v$ path in $\overline{G}$.

Hence $\overline{G}$ is connected.

Distance :

For any two points u, v of a graph we define the distance between u and v by

$$d(u,v) = \begin{cases} \text{the length of a shortest } u-v \text{ path if such a path exists} \\ \infty \text{ otherwise} \end{cases}$$

If G is a connected graph. $d(u,v)$ is always a non-negative integer.

Theorem: 4.7

A graph G with at least two points is bipartite iff all its cycles are of even length.

(49)

proof:

Suppose G is a bipartite. Then V can be partitioned into two subsets V_1 and V_2 such that every line joins a point of V_1 to a point of V_2.

Now, consider any cycle $v_0, v_1, v_2 \ldots v_n = v_0$ of length n.

Suppose $v_0 \in V_1$. Then $v_2, v_4, v_6 \ldots \in V_1$ and $v_1, v_3, v_5 \ldots \in V_2$.

$v_n = v_0 \in V_1$ and hence n is even.

Conversely, suppose all the cycles in G are of even length. We may assume without loss of generality that G is connected.

Let $v_1 \in V$. Define $V_1 = \{v \in V / d(v, v_1) \text{ is even}\}$

$V_2 = \{v \in V / d(v, v_1) \text{ is odd}\}$

clearly $V_1 \cap V_2 = \phi$ and $V_1 \cup V_2 = V$

We claim that every line of G joins a point of V_1 to a point of V_2

Suppose two points $u, v \in V_1$ are adjacent

Let P be a shortest $v_1 - u$ path of length m and let Q be a shortest $v_1 - v$ path of length n.

Since $u, v \in V_1$ both m and n are even.

Let u_1 be the last point common to P and Q.

Then the $v_1 - u_1$ path along P and $v_1 - u_1$ path along Q are both shortest paths and hence have the same length.

Now the $u_1 - u$ path along P, the line uv followed by the $v - u_1$ path along Q form a cycle of length $(m-i)+1+(n-i) = m+n-2i+1$ which is odd.

This is contradiction.

Thus no two points of V_1 are adjacent.

Similarly, no two points of V_2 are adjacent.

Hence G is bipartite.

Hence the theorem.

Cutpoint

A cutpoint of a graph G is a point whose removal increases the number of components.

1, 2, 3 are cutpoints

5 is non-cutpoint.

Bridge:

A bridge of a graph G is a line whose removal increases the number of components.

The lines {1, 2} and {3, 4} are bridges.

Theorem: 4.8

Let v be a point of a connected graph G. The following statements are equivalent.

1. v is a cut point of G
2. There exist a partition of V - {v} into subsets U and W such that for each u ∈ U and w ∈ W. The point v is on every u - w path.
3. There exist two points u and w distinct from v such that v is on every u - w path.

Proof.

① => ② Since v is a cut point of G

G - v is disconnected.

Hence G - v has at least two components.

let U consist of points of one of the component of G - v

W consist of the points of the remaining components clearly $V - \{v\} = U \cup W$ is the partition of $V - \{v\}$.

Let $u \in U$ and $w \in W$ Then u and w lie in different components of $G - v$

There is no $u-w$ path in $G-v$

Therefore every $u-w$ path in G contains v

(2) => (3) There exist two points u and w distinct from v such that v is on every $u-w$ path is trivial.

(3) => (1) Since v is on every $u-w$ path in G there is no $u-w$ path in $G-v$

Hence $G-v$ is not connected so that v is a cutpoint of G.

Theorem: 4.9

Let x be a line of a connected graph G. The following statements are equivalent.

1. x is bridge of G

2. There exist a partition of V into two subsets U and W such that for every point $u \in U$ and $w \in W$, the line x is on every $u-w$ path.

3. There exist two points u, w such that the line x is on every $u-w$ path.

proof:

(i) ⇒ (ii) Since x is bridge of G, $G-x$ is disconnected.

Hence $G-x$ has at least two components.

Let V consist of the points of one of the components of $G-x$ and W consist of the points of the remaining components.

Clearly $V = U \cup W$ is a partition of V.

Let $u \in U$ and $w \in W$ then u and w lie in different components of $G-x$. Hence there is no u-w path in $G-x$.

Therefore every u-w path in G contains x.

(ii) ⇒ (iii) There exist two points u, w such that the line x is on every u-w path is trivial.

(iii) ⇒ (i) Since x is on every u-w path in G there is no u-w path in $G-x$.

Hence $G-x$ is not connected so that x is bridge of G.

Theorem: 4.10

A line x of a connected graph G is a bridge iff x is not on any cycle of G.

Proof:

Let x be a bridge of G.

Suppose x lies on a cycle c of G.

Let w_1 and w_2 be any two points of G.

Since G is connected, there exist a $w_1 - w_2$ path p in G.

If x is not on p, then p is a path in $G-x$.

If x is on p, replacing x by $G-x$ we obtain a $w_1 - w_2$ walk in $G-x$.

This walk contains a $w_1 - w_2$ path in $G-x$. Hence $G-x$ is connected which is connected

Hence x is not on any cycle on G.

Conversely, let $x = uv$ be not on any cycle of G.

Suppose x is not a bridge.

Hence $G-x$ is connected.

There is a $u-v$ path in $G-x$

This path together with the line $x = uv$ forms a cycle containing x and this contradict

Hence x is a bridge.

Theorem: 4.11

Every non-trivial connected graphs

has at least two points which are not cutpoints.

proof:

choose two points u and v such that $d(u,v)$ is maximum.

we claim that u and v are not cutpoints.

Suppose v is a cutpoint.

Hence $G-v$ has more than one component.

choose a point w in a component that does not contain u.

Then v lies on every u-w path and hence $d(u,w) > d(u,v)$ which is impossible.

Hence v is not a cutpoint.

Similarly, u is not a cutpoint.

Hence the theorem.

Blocks:

A connected non-trivial graph having no cut point is a block. A block of a graph is a subgraph that is block and is maximal with respect to this property.

G

Blocks of G.

Theorem: 4.12

Let G be a connected graph with at least three points. The following statements are equivalent.

1. G is a block

2. Any two points of G lie on common cycle

3. Any point and any line of G lie on a common cycle.

4. Any two lines of G lie on a common cycle.

proof:

① => ② Suppose G is a block.

We shall prove by induction on the distance $d(u,v)$ between u and v that any two vertices u and v lie on a common cycle.

Suppose $d(u,v)=1$. Hence u and v are adjacent. By hypothesis $G \neq K_2$ and G has no cut points.

Hence the line $x = uv$ is not a bridge and hence by a line x of a connected graph G is a bridge iff x is not on any cycle of G

x is on cycle of G.

Hence the points u and v lie on common cycle of G.

Now assume that the result is true for any two vertices at distance less than k

Let $d(u,v) = k \geq 2$ consider u-v path of length k

Let w be the vertex that precedes v on this path

$d(u,w) = k-1$

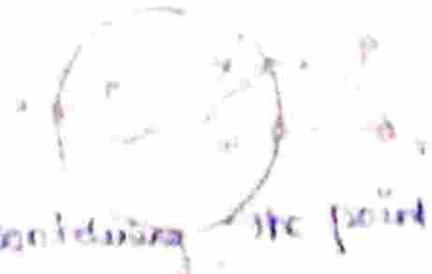
Hence by induction hypothesis there exist a cycle C that contains u and w. Now, since G is a block, w is not cutpoint of G

G - w is connected

Hence there exist a u-v path P not containing w

Let v' be the last point common to P and C such that since u is common to P and C

v' exists



Now, let Q denote the u-v' path along the cycle C not containing the point w

Then Q followed by the v'-v path along P the line vw and w-u path along the cycle C

has disjoint from Q forms a cycle that contains both u and v. This completes the induction

Thus any two points of G lie on a common cycle of G

(2) => (1) Suppose any two points of G lie on a common cycle of G

Suppose v is a cutpoint of G. Then there exist two point u and w distinct from v such that every u-w path contains v.

Now by hypothesis u and w lie on a common cycle. this cycle determines two u-w paths and at least one of these path does not contain v which is contradiction

Hence G has no cutpoints so that G is a block.

(2) => (3) let u be a point and vw a line of G. By hypothesis u and v lie on common cycle c

If u lies on c, the line vw together with the v-w path of c containing u is the required cycle containing u and the line vw.

If u is not on c. let c' be a cycle containing u and w

This cycle determines two $w-u$ paths and at least one of these paths does not contain v.

Denote this path by P.

Let u' be the first point common to P and C. Then the line vw followed by the $w-u'$ subpath of P and $u'-v$ path in C containing u and the line vw form a cycle containing u and the line vw.

③ => ② Any two points of G lie on a common cycle is trivial.

③ => ④ Any two lines of G lie on a common cycle is trivial.

④ => ③ Any point and any line of G lie on common cycle is trivial.

Connectivity:

The connectivity $k = k(G)$ of a graph G is the minimum number of points whose removal results in a disconnected or trivial graph.

Line connectivity: The line connectivity $\lambda = \lambda(G)$ of G is the minimum number of lines whose removal results in disconnected or trivial graph.

Example:

The connectivity and line connectivity of a disconnected graph is 0.

Theorem: 4.13

for any graph G, $k \leq \lambda \leq \delta$

proof:

We first prove $\lambda \leq \delta$. If G has no lines $\lambda = \delta = 0$. Otherwise removal of all the lines incident with a point of minimum degree results in a disconnected graph.

Hence $\lambda \leq \delta$

To prove $k \leq \lambda$, the following cases.

case i) G is disconnected or trivial. Then

$k = \lambda = 0$

Case ii) G is a connected graph with a bridge x. Then $\lambda = 1$.

$G = k_2$ or one of the points incident with x is a cutpoint

Hence $k = 1$ so that $k = \lambda = 1$

case iii) $\lambda \geq 2$. Then there exist λ lines the removal of which disconnects the graph

Hence the removal of $\lambda-1$ of these lines results in a graph G with a bridge $x = uv$.

For each of these $\lambda-1$ lines select an incident point different from u or v.

The removal of these $\lambda-1$ points removes all the $\lambda-1$ lines. If the resulting graph is disconnected, then $k \leq \lambda-1$

If not x is a bridge of this subgraph and hence the removal of u or v results in a disconnected or trivial graph.

Hence $k \leq \lambda$ and this completes the proof.

Remark:

$K = 2, \lambda = 3, \delta = 4$

n-connected:

A graph G is said to be n-connected if $k(G) \geq n$

n-line connected.

A graph G is said to be n-line connected if $\lambda(G) \geq n$

non-trivial graph is 1-connected iff G is connected

A non-trivial graph is 2 connected if it is block having more than one line. K_2 is the only block which is not 2-connected

Solved problems.

problem: 1

prove that if G is a k-connected graph then $q \geq \frac{pk}{2}$

Solution

Since G is k-connected, $k \leq \delta$

$$q = \frac{1}{2} \sum d(v)$$

$$\geq \frac{1}{2} p\delta \quad (\text{since } d(v) \geq \delta \text{ for } v)$$

$$\geq \frac{pk}{2}$$

problem: 2

prove that there is no 3-connected graph with 7 edges.

Solution.

Suppose G is a 3-connected graph with 7 edges

G has 7 edges $\Rightarrow p \geq 5$

$$q \geq \frac{3p}{2}$$

$$q \geq \frac{15}{2}$$

$q \geq 9$ which is contradiction

Hence there is no 3-connected graph

with 7 edges.

Eulerian and Hamiltonian graphs.

Eulerian graphs.

A closed trial containing all points and lines is called an Eulerian trail. A graph having an Eulerian trail is called an Eulerian graphs.

Example :

Lemma 5.1

If G is a graph in which the degree of every vertex is at least two then G contains a cycle.

proof

Construct a sequence $v, v_1, v_2 \ldots$ of vertices

Choose any vertex v. Let v_1 be any vertex adjacent to v

Let v_2 be any vertex adjacent to v_1 other than v

If vertex v_i, $i \geq 2$ is already chosen, then choose v_{i+1} to be any vertex adjacent to v_i other than v_{i-1}

Since degree of each vertex is at least 2, the existence of v_{i+1} is always guaranteed

Since G has only a finite number of vertices, at some stage we have to choose a vertex which has been choosen.

let v_k be the first such vertex

let $v_k = v_i$ $i < k$ Then

$v_i, v_{i+1} \cdots v_k$ is a cycle.

Theorem : 5.2

The following statements are equivalent for a connected graph of G

1. G is Eulerian

2. Every point of G has even degree

3. The set of edges of G can be partitioned into cycles.

proof :

(1) → (2) let T be an eulerian trail in G. with origin (and terminus) u.

Each time a vertex v occurs in T in a

place other than the origin and terminus, two of the edges incident with v are accounted for.

Since an eulerian trail contains every edges of G, d(v) is even for every $v \neq u$.

For u, one of the edges incident with u is accounted for by the origin of T another by the terminus of T and others are accounted for in pairs.

Hence d(u) is also even.

(2) => (3) Since G is connected and non trivial every vertex of G has degree at least 2. Hence G contains a cycle Z. The removal of the lines of Z result in a spanning subgraph G_1

In which again every vertex has even degree. If G_1 has no edges, then all the lines of G form one cycle.

Otherwise G_1 has a cycle Z_1

Removal of the lines of Z_1 from G_1 results in a spanning subgraph G_2 in which every vertex has even degree.

Continuing the above process when a graph G_n with no edge is obtained

We obtain a partition of the edges of G into n cycles

(3) => (1) If the partition has only one cycle then G is obviously Eulerian. Since it is connected.

Otherwise let $z_1, z_2 \ldots z_n$ be the cycles forming a partition of the lines of G.

Since G is connected there exist a cycle $z_i \neq z_1$ having a common point v_1 with z_1 without loss of generality, let it be z_2

The walk beginning at v_1 and consisting of the cycles z_1 and z_2 in succession is closed trail containing the edges of these two cycles.

Continuing this process, we can construct a closed trail containing all the edges of G

Hence G is eulerian

konigsberg bridge problem:

The graph of konigsberg bridges has vertices of odd degree

Hence it cannot have a closed trail running through every edge.

Hence one cannot walk through each of the Konigsberg bridges exactly once and come back to starting place.

Corollary 1:

Let G be a connected graph with exactly $2n$ $(n \geq 1)$ odd vertices. Then the edge set of G can be partitioned into n open trails.

proof:

Let the odd vertices of G can be labelled $v_1, v_2 \ldots v_n$; $w_1, w_2 \ldots w_n$ in any arbitrary order.

Add n edges to G between the vertex pairs (v_1, w_1) $(v_2, w_2) \ldots (v_n, w_n)$ to form a new graph G'

No two of these n edges are incident with the same vertex.

Every vertex of G' is of even degree and hence G' has an Eulerian trail T

If the n edges that we added to G are now removed from T. It will split into n open trails.

There are open trails in G and

form a partition of the edges of G

Corollary 2.

Let G be a connected graph with exactly two odd vertices. Then G has an open trail containing all the vertices and edges of G.

Proof.

Obviously the open trail mentioned begins at one of the odd vertices and end at the other.

Definition.

A graph is said to be arbitrarily traversable (traceable) from a vertex v if the following procedure always results in an eulerian trail

Start at v by traversing any incident edge on arriving at vertex u depart through any incident edge not yet traversed

continue until the lines are traversed.

If the graph is arbitrarily traversable from a vertex then it is

Arbitrarily Eulerian.

Theorem : 5.3

An Eulerian graph G is arbitrarily traversable from a vertex v in G iff every cycle in G contains v.

proof: Fleury's algorithm:

1. Choose an arbitrary vertex v_0 and set $W_0 = v_0$

2. Suppose that the trail $W_i = v_0 e_1 v_1 \ldots e_i v_i$ has been chosen. Then choose an edge e_{i+1} from $E(G) - \{e_1, e_2 \ldots e_i\}$ in such a way that

i) e_{i+1} is incident with v_i

ii) unless there is no alternative, e_{i+1} is not a bridge of $G - \{e_1, e_2 \ldots e_i\}$

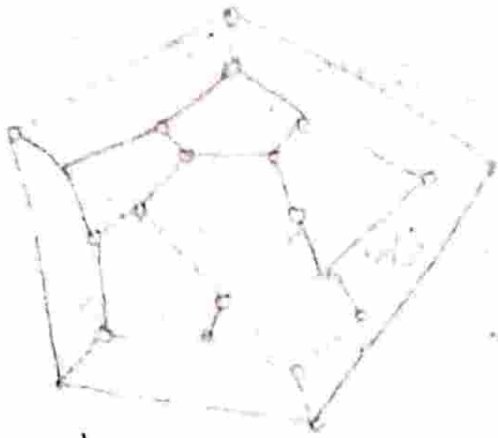
3. Stop when step 2 can no longer be implemented

Obviously, Fleury's algorithm construct a trail in G. It can be proved that if G is Eulerian, then any trail in G constructed by Fleury's algorithm is an Eulerian trail in G

Hamiltonian Graphs:

A spanning cycle in a graph is called a hamiltonian cycle.

A graph having a hamiltonian cycle is called a hamiltonian graph.



Theta graph:

A block with two non-adjacent vertices of degree 3 and all other vertices of degree 2 is called theta graph.

Theta graph consists of two vertices of degree 3 and three disjoint paths joining them.

Each of length at least 2.

The theta graph is non-hamiltonian and every non hamiltonian 2 connected graph has a theta subgraph.

Theorem: 5.4

Every hamiltonian graph is 2-connected

proof:

Let G be a hamiltonian graph.

Let z be a hamiltonian cycle in G.

For any vertex v of G, $z-v$ is connected

and hence $G-v$ is also connected.

Hence G has no cutpoints and thus

G is 2-connected.

Theorem: 5.5

If G is hamiltonian, then for every

non-empty proper subset S of $V(G)$,

$\omega(G-S) \leq |S|$ where $\omega(H)$ denotes the

number of components in any graph H.

Proof:

Let z be a hamiltonian graph cycle

of G. Let S be any non-empty proper

subset of $V(G)$

Now, $\omega(z-S) \leq |S|$. Also $z-S$ is a spanning subgraph of $G - S$

Hence $w(G - S) \leq w(Z - S)$

Hence $w(G - S) \leq |S|$.

Theorem : 5.6 (Dirac 1952). If G is a graph with $p \geq 3$ vertices and $\delta \geq p/2$ then G is hamiltonian.

proof:

Suppose the theorem is false.

Let G be a maximal (with respect to number of edges) non hamiltonian graph with p vertices and $\delta \geq p/2$

Since $p \geq 3$, G cannot be complete.

Let u and v be non-adjacent vertices in G.

By the choice of G, $G + uv$ is hamiltonian. Moreover, since G is non-hamiltonian

Each hamiltonian cycle of $G + uv$ must contain the line uv.

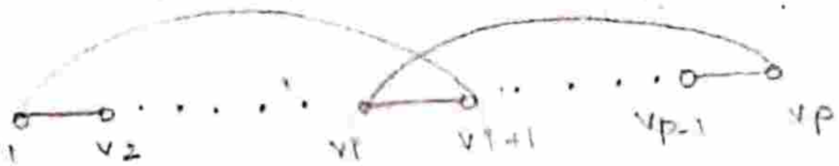
Thus G has a spanning path $v_1, v_2 \ldots v_p$ with origin $u = v_1$ and terminus $v = v_p$

Let $S = \{ v_i \mid uv_{i+1} \in E \}$ and

$T = \{ v_i \mid i < p \text{ and } v_i v \in E \}$ where E is the edge set of G.

Clearly $v_p \notin S \cup T$ and hence $|S \cup T| < p$. ①

Again if $v_i \in S \cap T$, then $v_1, v_2 \ldots v_i v_p v_{p-1} \ldots v_{i+1} v_1$ is a hamiltonian cycle in G.



Hence $S \cap T = \phi$ so that $|S \cap T| = 0$ .. ②

By the definition of S and T

$$d(u) = |S| \qquad d(v) = |T|$$

① and ② $\Rightarrow$ $d(u) + d(v) = |S| + |T|$

$$= |S \cup T| < p$$

$d(u) + d(v) < p$

$\delta \geq p/2$, $d(u) + d(v) \geq p$ which gives the contradiction

Hence the theorem.

Lemma 5.7

Let G be a graph with p points and let u and v be non-adjacent points in G such that $d(u) + d(v) \geq p$. Then G is hamiltonian iff $G + uv$ is hamiltonian.

proof:

If G is hamiltonian, then obviously $G+uv$ is also hamiltonian.

Conversely, suppose that $G+uv$ is hamiltonian but G is not.

Then, we obtain $d(u)+d(v) < p$

This contradicts the hypothesis that $d(u)+d(v) \geq p$

Thus $G+uv$ is hamiltonian implies G is hamiltonian.

Closure

The closure of a graph G with p points is the graph obtained from G by repeatedly joining pairs of non-adjacent vertices whose degree sum is at least p until no such pair remains. The closure of G is denoted by $C(G)$. 8's vertices

Theorem: 6.8 $C(G)$ is well defined.

proof:

let G have p vertices.

Let G_1 and G_2 be two graphs obtained from G by repeatedly joining pairs of non-adjacent vertices

whose degree sum is at least p until no such pair remains.

Let $x_1, x_2 \ldots x_m$ and $y_1, y_2 \ldots y_n$ be the sequences of edges added to G in obtaining G_1 and G_2

We claim that $\{x_1, x_2 \ldots x_m\} = \{y_1, y_2 \ldots y_n\}$

If possible, let $x_{i+1} = uv$ be the first edge in the sequence $\{x_1, x_2 \ldots x_m\}$ that is not an edge of G_2.

Let $H = G + \{x_1, x_2 \ldots x_i\}$ since uv is the next edge to be added to H in the process of constructing G_1, we have

$$d_H(u) + d_H(v) \geq p \quad \ldots (1)$$

By the choice of x_{i+1}, H is the subgraph of G_2.

Hence $d'(u) \geq d_H(u)$ and $d'(v) \geq d_H(v)$ where $d'(u)$ and $d'(v)$ denote the degrees of u and v in G_2

$$d'(u) + d'(v) \geq p.$$

By the definition of G_2, u and v must be adjacent in G_2. This is Contradiction.

Since u and v are not adjacent in G_2

Each x_i is an edge of G_2.

Similarly, we can be prove that each y_i

is an Edge of G_1

Hence $G_1 = G_2$

$C(G)$ is unique and hence is well defined

Theorem: 5.9

A graph is hamiltonian iff its closure is hamiltonian.

Proof:

let $x_1, x_2 \ldots x_n$ be the sequence of edges added to G in obtaining $C(G)$.

Let $G_1, G_2 \ldots G_n = C(G)$ be the successive graphs obtained (Lemma 5.7)

G is hamiltonian $\Leftrightarrow$ G_1 is hamiltonian

$\Leftrightarrow$ G_2 is hamiltonian

$\vdots$

$\Leftrightarrow$ $G_n = C(G)$ is hamiltonian

Theorem: 5.10 (Chavatal, 1972). Let G be a graph with degree sequence $(d_1, d_2 \ldots d_p)$ where $d_1 \le d_2 \le \ldots \le d_p$ and $p \ge 3$

Suppose that for every value of m less than $\frac{p}{2}$, either $d_m > m$ or $d_{p-m} > p-m$ (ie, there is no value of m less than $p/2$ for which $d_m \leq m$ and $d_{p-m} < p-m$). Then G is hamiltonian.

Proof:

Let G satisfy the hypothesis of theorem. We claim that $C(G)$ is complete.

Let us denote the degree of vertex v in $C(G)$ by $d'(v)$

If possible, Let $C(G)$ be not complete.

Now, let u and v be two non-adjacent vertices in $C(G)$ with

$$d'(u) \leq d'(v) \quad \ldots (1)$$

and $d'(u) + d'(v)$ as large as possible.

Let $d'(u) = m$. Since no two non-adjacent points in $C(G)$ can have degree sum p or more

We have $d'(u) + d'(v) < p$

$$d'(v) < p - d'(u)$$

$$d'(v) < p - m \quad \ldots (2)$$

Now, let S denote the set of vertices in $V - \{v\}$ which are not adjacent to v in $C(G)$

Let T denote the set of vertices in $V - \{u\}$ which are not adjacent to u in $C(G)$

Clearly, $|S| = p-1-d'(v)$

$$|T| = p-1-d'(u) \quad \cdots \quad (3)$$

By the choice of u and v, each vertex in S has degree at most $d'(u)$

Each vertex in $T \cup \{u\}$ has degree at most $d'(v)$

(2) in (5) $\Rightarrow$ $|S| > p-1-(p-m) = m-1$

Hence $|S| \geq m$

$C(G)$ has at least m points with degree $\leq m$.

(5) $\Rightarrow$ $|T| = p-1-m$. Since each vertex in $T \cup \{u\}$ has degree $\leq d'(v)$ This implies that $C(G)$ has at least $p-m$ vertices of degree $\leq d'(v)$

By (2) $C(G)$ has atleast $p-m$ vertices of degree $< p-m$.

Because G is a spanning subgraph of $C(G)$ degree of each point in G cannot exceed that in $C(G)$

Hence $d_m < m$ and $d_{p-m} < p-m$.

(1) and (2) $\Rightarrow$ $m < p/2$

This contradicts the hypothesis on G.

$C(G)$ is complete. Hence G is hamiltonian.

Solved problem:

problem 1:

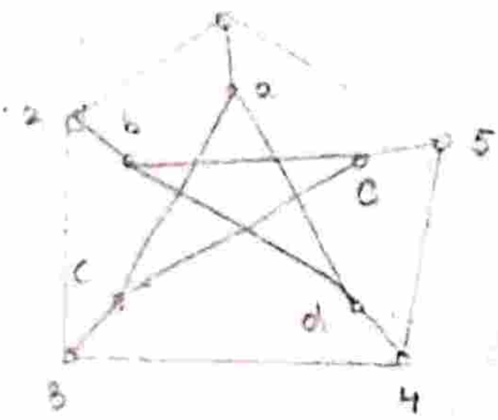
Show that the petersen graph is non-hamiltonian.

Solution:

If the petersen graph G has a hamiltonian cycle C. then $G - E(C)$ must be regular spanning subgraph of degree 1.

(A regular spanning subgraph of degree 1 is called 1-factor).

Let us search for all 1-factors in G. and show that none of them arise out of a hamiltonian cycle of G.



Case 1.

Consider the subset A = {1a, 2b, 3c, 4d, 5e}

Of the edge set of G

Clearly A is a 1-factor of G

But G−A is the union of two disjoint

cycles

Hence is not hamiltonian cycle of G.

Case 2 :

If the 1-factor contains 4 edges from A

then the only line passing through the

remaining two points must also be

included in the 1-factor

So that we again get A

Case 3 : If a 1-factor contains just 3 edges

from A, there two such choices can be made

Subcase 3 A :

Let the 1-factor Contains 1a, 2b and 3c

Now the subgraph induced by the remaining

four points is P_4 whose unique 1-factor is

{4d, 5e}.

Thus the 1-factor of G Considered

becomes A

Subcase 3B :

Let the 1-factor Contain 1a, 2b and 4d.

Here again the remaining four points induces P_4

whose unique 1-factor is {3c, 5e}

Thus the 1-factor of G considered becomes A

Case 4 : If a 1-factor contains just 2 edges from A, then again two such choices are possible.

Subcase 4A :

Let the 1-factor contain 1a and 2b

In the subgraph induced by the remaining 6 points point d has degree one

Hence any 1-factor of that subgraph must contain edge 4d. The case 3 is repeated.

Subcase 4B :

Let the 1-factor Contain 1a and 3b.

In the subgraph induced by the remaining 6 points.

point 2 has degree one and hence any 1-factor. of that Subgraph Must contain edge 2b Thus case 3 is repeated.

Case 5.

Let a 1-factor contain just one edge of A say 1a.

If it contain one more edge from A then one of the earlier cases will be repeated.

Hence we have to choose the other four edges of this 1-factor from two paths each of length 3.

Hence the 1-factor is B = {1a, ce, bd, 23, 45}. Now G − B is again union of two disjoint cycle and not a hamiltonian cycle.

Case 6:

Suppose there exist a 1-factor that does not contain any edge from A.

It can contain at most two edges from the cycle 1 2 3 4 5 1 and at most two edges from the cycle a c e b d a. Hence it can contain at most four edges.

Hence there does not exist such 1-factor.

Since the above 6 cases cover all possible types of 1-factors we see that G has no 1-factor resulting out of hamiltonian cycle.

G has no hamiltonian cycle.

G is not hamiltonian.

Trees:

A gr...

called an a...

A ...

a tree ...

called a ...

of tree ...

Theorem : ...

following ...

1. ...

2. ...

by a uni...

3. ...

4. ...

proof

① ⇒ ② ...

Given ...

Proves ...

Unit - 3

Trees:

A graph that contains no cycles is called an acyclic graph.

A connected acyclic graph is called a tree. Any graph without cycles is also called a forest. So that the component of forest are trees.

Theorem: 6.1

Let G be a (p, q) graph. The following statements are equivalent.

1. G is a tree
2. Every two points of G are joined by a unique path
3. G is connected and $p = q+1$
4. G is acyclic and $p = q+1$

proof:

1 => 2 Let u, v be any two points in G. Since G is connected, there exist a $u-v$ path in G. suppose there exist two distinct

$P_2 : u = w_0, w_1, w_2 \cdots w_m = v$

Let i be the least positive integer such that $1 \le i < m$ and $w_i \notin P_1$

Hence $w_{i-1} \in P_1 \cap P_2$

Let j be the least positive integer such that $i < j \le m$ and $w_j \in P_1$. Then the $w_{i-1} - w_j$ path along P_2 followed by the $w_j w_{i-1}$ path along P_1 form a cycle which is contradiction

Hence there exist a unique u-v path in G.

(2) $\Rightarrow$ (3) Clearly G is connected.

We prove $p = q+1$ by induction on p

This is trivial for connected graph with 1 or 2 points.

Assume the result for graphs with fewer than p points.

Let G be graph with p points.

Let $x = uv$ be any line of G.

Since there exist a unique u-v path in G. $G - x$ is disconnected graph with exactly two components G_1 and G_2

Let G_1 be (p_1, q_1) graph and G_2 be (p_2, q_2) graph.

Then $p_1 + p_2 = p$ and $q_1 + q_2 = q - 1$

$p_1 = q_1 + 1 \qquad p_2 = q_2 + 1$

$$p = p_1 + p_2$$
$$= q_1 + q_2 + 2$$
$$= q + 1$$

(3) => (4) We must prove that G is acyclic.

Suppose G contains a cycle of length n

There are n points and n lines on this cycle. Fix a point u on the cycle.

consider any one of the remaining p-n points not on the cycle say v.

Since G is connected we can find a shortest u-v path in G.

Consider the line on this shortest u-v path in G and incident with v.

The p-n lines thus obtained are all distinct.

Hence $q \geq (p-n)+n = p$ which is contradiction.

Since $q+1 = p$. Thus G is acyclic.

(4) => (1). Since G is acyclic to prove that G is a tree we need only to prove that G is connected.

Suppose G is not Connected. Let

$G_1, G_2, \ldots G_k$ $(k \geq 2)$ be the components of G

Since G is acyclic each of these components is a tree

Hence $q_i + 1 = p_i$ where G_i is a (p_i, q_i) graph

$$\sum_{i=1}^{k} (q_i + 1) = \sum_{i=1}^{k} p_i$$

$q + k = p$ and $k \geq 2$ which is contradiction

Hence G is connected

This completes the proof

Corollary: Every non-trivial tree G has at least two vertices of degree 1.

Proof:

- Since G is non-trivial

$d(v) \geq 1$ for all point v

$\sum d(v) = 2q = 2(p-1) = 2p - 2$

$d(v) = 1$ for at least two vertices.

Theorem 6.2 Every connected graph has a spanning tree.

Proof:

Let G be a connected graph

Let T be a minimal connected spanning subgraph of G

Then for any line x of T

$T-x$ is disconnected and hence

x is a bridge of T

Hence T is acyclic.

Further T is connected and hence is a tree.

Corollary: Let G be a (p, q) connected graph.

Then $q \geq p-1$.

proof: Let T be a spanning tree of G

Then the number of lines in T is $p-1$

Hence $q \geq p-1$

Theorem: 6.3

Let T be a spanning tree of a connected graph G. Let $x = uv$ be an edge of G not in T. Then $T+x$ contains a unique cycle.

proof: Since T is acyclic every cycle in $T+x$ must contain x.

Hence there exist a one to one correspondence between cycles in $T+x$ and $u-v$ paths in T.

As there is a unique $u-v$ path in tree T, there is a unique cycle in $T+x$.

Centre of a tree :

Definition :

Let v be a point in connected graph G.

Eccentricity : The eccentricity of $e(v)$ of v is defined by $e(v) = \max\{d(u, v) \mid u \in V(G)\}$.

Radius : The radius $r(G)$ is defined by

$r(G) = \min\{e(v) \mid v \in V(G)\}$

Central point : v is called a central point if $e(v) = r(G)$

Centre : The set of all central points is called the centre of G.

Theorem : 6.4

Every tree has a centre consisting either one point or two adjacent points.

proof:

The result is obvious for the trees K_1 and K_2.

Now, let T be any tree with $p \geq 2$ points.

T has at least two end points and maximum distance from a given point u to any other point v v is an end point.

Now, delete all the end points from T.

The resulting graph T' is also a tree

The Eccentricity of each point in T' exactly

or less than the eccentricity of same

point in T

Hence T and T' have same centre.

If the process of removing end points

is repeated.

We obtain successive trees having the

same centres as T

We eventually obtain a tree which

is either K_1 or K_2.

Hence the centre of T consist of either

one point or two adjacent points.

Matchings:

Any set M of independent lines of

a graph G is called a matching of G.

We say that u and v are matched

under M.

M-Saturated:

The points u and v are M-Saturated.

Perfect Matching:

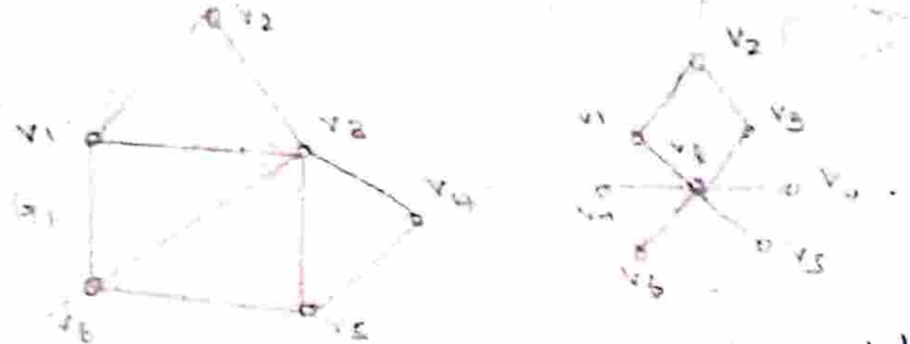
A matching M is called a perfect Matching if every point of G is M-saturated

Maximum matching:

M is called a maximum matching if there is no matching M' in G such that

$$|M'| > |M|$$

Example:



In G_1

$M_1 = \{v_1v_2, v_6v_3, v_5v_4\}$ is perfect matching in G_1

$M_2 = \{v_1v_3, v_6v_5\}$ is matching in G_2

M_2 is not perfect matching.

The points v_2 and v_4 are not M_2-Saturated

In G_2 $M = \{v_8v_4, v_1v_2\}$ is maximum matching but is not a perfect matching.

M-alternating path:

Let M be a matching in G. A path in G is called an M-alternating path.

If the lines are alternately in X-M and M.

Example:

In $G_1 \Rightarrow P_1 = (v_6, v_5, v_4, v_3)$ is an M_1-alternating path.

In $G_2 \Rightarrow (v_7, v_9, v_4)$ is an M-alternating path $G_1: M_1 \Delta M_2 = \{v_1v_2, v_6v_3, v_5v_4, v_1v_3, v_6v_5\}$.

M-augmenting path:

An M-alternating path whose origin and terminus are both M-unsaturated is called an M-augmenting path.

Example:

In $G_1 \Rightarrow P_2 = (v_2, v_1, v_3, v_6, v_5, v_4)$ is an M-augmenting path.

Remark:

$p = 2|M|$ and p is even. The graph G_2 has an even number of vertices but has no perfect Matching.

Theorem: 7.1

Let M_1 and M_2 be two matching in a graph G. Let $M_1 \Delta M_2 = (M_1 - M_2) \cup (M_2 - M_1)$ be the Symmetric difference of M_1 and M_2. Let $H = G[M_1 \Delta M_2]$ be subgraph of G induced by $M_1 \Delta M_2$. Then each component of H is either an even cycle with edges alternately in M_1 and M_2 or a path p with edges alternately in M_1 and M_2 such that the origin and the terminus of p are unsaturated in M_1 or M_2.

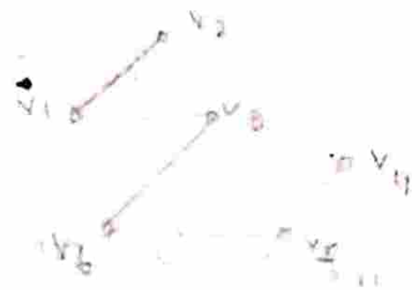
Proof: Let v be any point in H

Since M_1 and M_2 are matchings in G, at most one line of M_1 and at most one line of M_2 are incident with v

Hence the degree of v in H is either 1 or 2.

Hence it follows that the components of H must be described in theorem.

Example:



The graph $H_1 = G_1 [M_1 \Delta M_2]$

H_1 is path whose edges are alternately in M_1 or M_2.

The origin v_2 and terminus V_4 are both M_2 - unsaturated.

Theorem: T.2 A matching M in a graph G is a maximum matching if and only if

G contains no M- augmenting path.

proof: Let M be maximum matching in G

Suppose G contains M - argumenting path

$P = (v_0, v_1, v_2 \ldots v_{2k+1})$

By definition of M- augmenting path the line $v_0v_1, v_2v_3 \ldots v_{2k}v_{2k+1}$ are not in M.

The lines $v_1v_2, v_3v_4 \ldots v_{2k-1}v_{2k}$ are in M.

$$M' = M - \{v_1v_2, v_3v_4 \ldots v_{2k-1}v_{2k}\} \cup \{v_0v_1, v_2v_3, \ldots v_{2k}v_{2k+1}\}$$

is a matching in G

$|M'| = |M| + 1$ which is contradiction.

Since M is maximum matching.

Hence G has no M - argumenting path.

Conversely, suppose G has no M - argumenting path.

If M is not a maximum matching in G, then there exist a matching M' of G

Such that $|M'| > |M|$

Let $H = G[M \Delta M']$. Each component of H is either an even cycle with edges alternately in M and M' or a path P with edges alternately in M and M'

Such that the origin and terminus of P are unsaturated in M or M'.

Clearly any component of H which is a cycle contains equal number of edges from M and M'.

Since $|M'| > |M|$ there exist at least one component of H

which is a path whose first and last edges are from M'.

Thus the origin and terminus of P are M'-saturated in H and hence they are M-unsaturated in G.

P is an M-augmenting path in G

which is contradiction.

Hence M is a maximum matching in G.

Solved problem:

problem 1:

For what values of n does the complete graph K_n have perfect matching.

Solution:

Clearly any graph with p odd has no perfect matching.

Also the complete graph K_n has a perfect matching if n is even.

For example if $V(K_n) = \{1, 2, \ldots n\}$ then $\{12, 34, \ldots (n-1)n\}$ is a perfect matching of K_n

Thus K_n has a perfect matching if and only if n is even.

problem: 2
Show that a tree has at most one perfect matching.

solution: let T be a tree

Suppose T has two perfect matchings say M_1 and M_2.

Then degree of every vertex in $H = T[M_1 \Delta M_2]$ is 2.

$\therefore$ Hence every component of H is an even cycle with edges alternately in M_1 and M_2.

This is contradiction, since T has no cycles. Therefore T has at most one perfect matching

problem: 3
Find the number of perfect matching in the complete bipartite graph $K_{n,n}$.

Solution :

Let $A = \{x_1, x_2 .. x_n\}$ $B = \{y_1, y_2 .. y_n\}$ be bipartition of $K_{n,n}$

We observe that any matching of $K_{n,n}$ that saturates every vertex of A is a perfect matching.

Now the vertex x_1 can be saturated in n ways by choosing any one of the edges $x_1y_1, x_1y_2 \ldots x_1y_n$.

Having saturated x_1, the vertex x_2. Can be saturated in $n-1$ ways.

In general, having saturated $x_1, x_2 .. x_i$, the next vertex x_{i+1} can be saturated in $n-i$ ways.

The number of perfect matching in $K_{n,n}$ is $n(n-1) \ldots 2 \cdot 1 = n!$

problem : 4 find the number of perfect matchings in the complete graph K_{2n}.

Solution :

Let $V(K_{2n}) = \{v_1, v_2 .. v_{2n}\}$.

The vertex v_1 can be saturated in $2n-1$ ways by choosing any line e_1 is incident at v_1.

In general having chosen the edges $e_1, e_2 \cdots e_k$, a vertex v which is not saturated.

By any of the edges $e_1, e_2 \ldots e_k$ can be saturated in $2n-(2k+1)$ ways.

We obtain a perfect matching after the choice of n lines in the above process.

The number of perfect matchings in K_{2n}

$$= 1 \cdot 3 \cdot 5 \cdots (2n-1)$$

$$= \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdots (2n-1)(2n)}{2 \cdot 4 \cdot 6 \cdots 2n}$$

$$= \frac{(2n)!}{2^n n!}$$

Matchings in bipartite graphs:

Neighbour Set:

For a subset S of V the neighbour set $N(S)$ is the set of all points each of which is adjacent to at least one vertex in S

Theorem: 7.3 (Hall's marriage theorem).

Let G be a bipartite graph with bipartition (A, B). Then G has a matching that

Saturates all the vertices of A if and only if $|N(S)| \geq |S|$, for every subset S of A.

Proof:

Suppose G has a matching M that saturates all the vertices in A.

Let $S \subseteq A$; Then every vertex in S is matched under M to a vertex in N(S) and two distinct vertices of S are matched to two distinct vertices of N(S)

$|N(S)| \geq |S|$.

Conversely, suppose $|N(S)| \geq |S|$ for all $S \subseteq A$. We show that G contains a matching which saturates every vertex in A.

Suppose G has no such matching.

Let M^* be a maximum matching in G, By assumption there exists a vertex $x_0 \in A$ which is M^*- unsaturated.

Let $Z = \{v \in V(G) \mid$ there exists a M^*- alternating path connecting x_0 and $v\}$

Since M^* is a maximum matching,

By berge's theorem,

G has no M^*- augmenting path

Hence x_0 is the only M^*- unsaturated vertex in Z.

Let $S = Z \cap A$ and $T = Z \cap B$. clearly $x_0 \in S$ and every vertex of $S - \{x_0\}$ is matched under M^* with a vertex of T

$$|T| = |S| - 1 \quad \ldots (2)$$

We claim that $N(S) = T$, By definition of T

$$T \subseteq N(S) \quad \ldots (3)$$

Let $v \in N(S)$ there exists $u \in S$ such that v is adjacent to u.

$$S = Z \cap A \qquad u \in Z$$

Hence there exists M^*- alternating path P

$$(x_0, y_1, x_1, y_2 \ldots x_{k-1}, y_k, u)$$

If v lies on P, then clearly $v \in Z \cap B = T$

Suppose v does not lie on P.

Now the edge $y_k u \in M^*$. Hence the edge uv is not in M^*

Hence the path P, consisting of P followed by edge uv is an M^*- alternating path. $v \in Z \cap B = T$ $\quad N(S) \subseteq T \quad \ldots (4)$

From ③ and ④

$$N(S) = T \quad \ldots ⑤$$

② and ⑤ $|N(S)| = |T| = |S| - 1 < |S|$

Which is contradiction

Hence the theorem.

Theorem: 7.4

Let G be a k-regular bipartite graph with $k > 0$. Then G has a perfect matching.

proof:

Let (V_1, V_2) be bipartition of G.

Since each edge of G has one end in V_1 and there are k-edges incident with each vertex of V_1

$$q = k|V_1|$$

By similar argument, $q = k|V_2|$

$$k|V_1| = k|V_2|$$

Since $k > 0$ we get $|V_1| = |V_2|$

Now, let $S \subseteq V_1$. let E_1 denote the set of all edges incident with vertices in $N(S)$

Since G is k-regular,

$$|E_1| = k|S| \text{ and } |E_2| = k|N(S)|$$

By definition of $N(S)$

$$E_1 \subseteq E_2$$

$$k|S| \leq k|N(S)|$$

$$|N(S)| \geq |S|$$

By hall's theorem, G has a matching M that saturates every vertex in V_1

$$|V_1| = |V_2|$$

M also saturates all the vertices of V_2

M is a perfect matching.

Unit - 4

III - Year V - Semester

course code: 7BMAE1A

Elective course: I(A) - Graph theory.

Unit - I

Graphs - Definition of Examples Degrees - Sub graphs - Isomorphism - Ramsey Numbers - Independent sets & coverings - Intersection graphs and line graphs - matrices - operations co graphs.

Unit - II

Degree sequences - Graphic sequences - walks, Trials and paths - connectedness and components - Blocks connectivity - Eulerian graphs - Hamiltonian Graphs.

Unit - III

Trees - Characterisation of trees - centre of a tree - matching - matching in Bipartite graphs.

Unit - IV

planer graphs and properties - characterization of planer graphs - Thickness. Crossing and outer planarity - charact chromatic number and chromatic Index - the fine colour theorem & four colour Problem

Unit - V

Chromatic polynomials - Definition and basic propeities of Directed Graph - paths & connections - Diagraphs and matices - Tournaments.

Text Book:

1. Iniciation to Graph theory by Dr. S Arumugam & S. Ramachandran Scitech Publication Pvt. Ltd 2001.

Unit I Chapter 2

Unit II Chapter 3, 4, 5

Unit III Chapter 6, 7

Unit IV Chapter 8, 9.1, 9.3

Unit V Chapter 9, 9.4, 10

Unit-4

Planarity.

Definition and Properties.

A graph is said to be embedded in a surface S when it is drawn on S so that no two edges intersect ("meeting" of edges at a vertex is not considered an intersection) A graph is called planar if it can be drawn on a plane without intersecting edges. A graph is called non-planar if it is not planar. A graph that is drawn on the plane without intersection edges is called a plane graph.

Example: The graph is Fig 8.1(a) is planar eventhough it is not Plane

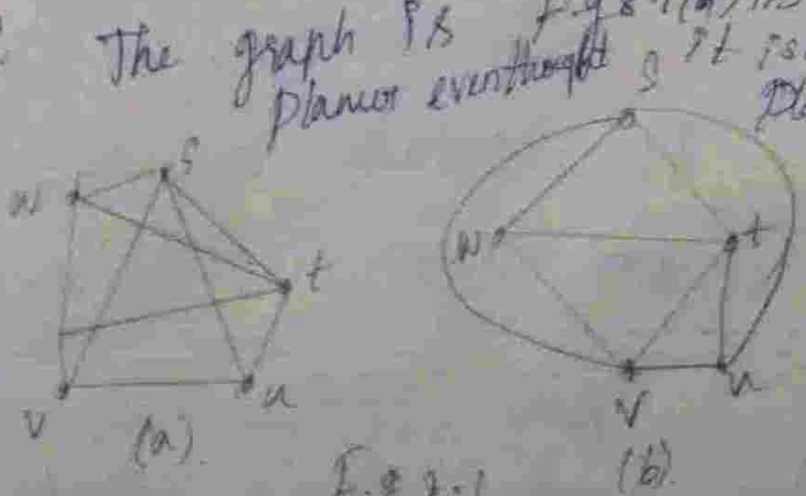


Fig 8.1

The graph 8.1(b) (which is isomorphic to that Fig (8.1)(a)) is plane as it is drawn without intersecting edges. It is also planner. Thus plane graph is a concept associated with embedding of the graph.

It is obvious that if two graphs are isomorphic and one is planner. Then the other is also planer. However, as is seen from 8.1. If two graphs are isomorphic and one is plane. the other need not be plane.

Thm 8.1.

K_5 is non- planer.

Proof:

If possible, let K_5 be planar. K_5 contains a cycle of length five say (, t, u, v, w, s).

Hence, without loss generality, any plane embedding of K_5

can be assumed to contain this cycle drawn in the form of a regular pentagon (see Fig 8.1(b)). Hence the edge wt must lie either wholly inside the pentagon or wholly contain it.

Suppose that wt is wholly inside the pentagon. (The argument when it is wholly outside the pentagon is quite similar). Since the edge sv and su do cross the edge wt, they must both the outside the pentagon. The edge vt cannot cross the edge su. Hence vt must be inside the pentagon. But now, the edge uw crosses one of the edges already drawn, giving a contradiction. Hence K is a non- planner.

Definition

Let G be a graph embedded on a plane Π. Then Π - G is union of disjoint regions. such regions are called faces of G. Each plane graph has exactly one unbounded face and it is called the exterior face. Let F is a face of a plane graph G and e be an edge of G. Let p be a point in ______ said to be in the boundary of F if for every point Q of Π on e there exist curve joining p and Q which lies entirely in F.

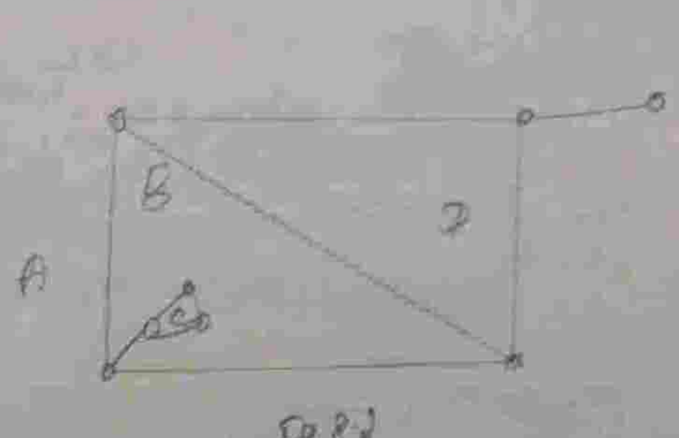


Fig 8.2

For the plane graph in Fig 8.2 A, B, C and D are the faces.

They have 5 and 3 edges respectively in their boundary. A is the exterior face of G.

Thm 8-2

A graph can be embedded in the surface of a sphere iff it can be embedded in a plane.

Proof:

Let G be a graph embedded on a sphere. Place the sphere on a plane and call the point of plane I. and call the point of contact S (South pole). At point S, draw a normal the plane and let N (North pole). be the point where this normal intersects the surface of the sphere.

Assume that the sphere is placed in such a way that N is disjoint from G.

For each point p on the sphere, let $\bar{p}$ be the unique point on the plane where the line NP intersects the surface of the plane. Thus there is a one-to-1 correspondence between the points of the sphere other than N and the point on the plane.

(P' is called the sterographic projection of P on L)

In this way, the vertices and the edges of G can be projected on the plane L, which gives an embedding of G in L.

The reverse process obviously gives an embedding in the sphere for any graph that is embedded in the plane L. This completes the proof.

Thm 8.3:

Every 2-connected plane graph can be embedded in the plane so that any specified face is the exterior face.

Proof:

Let F be a nonexterior face of a plane 2-connected graph G. Embed G on a sphere and call some point interior to F as the north pole. Consider a plane tangential to the sphere at the south pole and project (stereographic projection). G onto that plane from the north pole. The result is a plane embedding of G. Since $N \in F$, the image of F under this projection is the unbounded face (exterior face) of this plane embedding.

We state the following theorem without giving its proof.

Thm 8.4

(Fary, 1948). Every planar graph can be embedded in a plane such that all edges are straight line segments.

Definition:

A graph is polyhedral (in fact 3- polyhedral) if its vertices and edges may be identified with the vertices and edges of a convex polyhedron in three dimensions.

A graph is polyhedral iff it is planar and 3- connected.

Thm 8.5.

Every polyhedron has at least two faces with the same number of edges on the boundary.

Proof:

The corresponding graph G is 3- connected. Hence $\delta(G) \geq 3$ and the number of faces adjacent to any chosen face f is equal to the number of edges in the boundary of face f. (If two faces have the edges vu and vw with $v \neq w$ in common, then $G - \{v, w\}$ is disconnected contradicting 3- connectedness). Let

$f_1, f_2, \dots f_m$ be the faces of the polyhedron and e_i be the number of edges on the boundary of the i^{th} face. let the faces be labelled so that $e_i \leq e_{i+1}$ for every

If no two faces have the same number of edges in their boundaries that $e_{i+1} = e_i \geq 1$ for every i.

Hence so that $e_m \geq e_1 + m - 1$.

Since $e_1 \geq 3$, this implies that $e_m \geq m+2$ so that the m^{th} face is adjacent to atleast $m+2$ faces. This gives a contradiction as there are only m for This proves the theorem.

The Following result is often called Euler's polyhedron formula since it relavant the number of vertices. edges and faces of a convex polyhedron.

Thm 8.6. (Euler). If G is a connected plane graph having V, E and as the sets of vertices, edges and faces respectively, then $|V| - |E| + |F| = 2$.

Proof:

The proof is by induction on the number of edges of G.

Let $|E| = 0$

Since G is connected, it is K_1 so that $|V| = 1$, $|F| = 1$ (the infinite face). as hence $|V| - |E| + |F| = 2$.

Now let G be a graph as in the theorem and suppose that the theorem is true for all connected plane graphs with at most $|E| - 1$ edges.

If G is a tree, then $|E| = |V| - 1$ and $|F| = 1$ and hence $|V| - |E| + |F| = 2$.

G is not a tree, let x be an edge contained in some cycle of G.

Then $G' = G - x$ is a connected plane graph such that $|V(G')| = |V|$,

$|E(G')| = |E| - 1$.

N: $|F(G')| = |F|-1$.

Hence by the induction hypothesis

$|N(G')| - |E(G')| + |F(G')| = 2$ so that

$|V| - (|E|-1) + |F|-1 = 2$

Hence $|V| - |E| + |F| = 2$ as required

This complete the induction and the proof.

Corollary 1: If G is a plane (p,q) graph with r faces and k components the $p-q+r = k+1$.

Proof.

consider a plane embedding of G such that the exterior face of $-$ component contains all other components. Now let ith component be (p,q) graph with r_i faces for each i. By the theorem

$p_i - q_i + r_i = 2$.

Hence $\Sigma p_i - \Sigma q_i + \Sigma r_i = 2k \ldots (1)$

But $\Sigma p_i = p$, Σq_i and $\Sigma r_i = r+(k-1)$

since the infinite face is ... k times in Σr_i

Hence (1) gives $p-q+r+k-1=2k$

so that $p-q+r=k+1$ as required

Corollary 2: If G is a (p,q) plane graph in which every face is an n cycle then $q=\dfrac{n(p-2)}{n-2}$

Proof:

Every face is an n-cycle. Hence ~~...~~ each edge lies on the boundary of exactly two faces. Let $f_1, f_2, \ldots, f_r$ be the faces of G.

$2q = \sum_{i=1}^{r}$ (number of edges in the boundary of face f_i) $= nr$

$r = 2q/n$

By Euler's formula $p-q+r=2$.

$$\therefore p-q+\frac{2q}{n}=2$$

$$q\left(2/n-1\right)=2-p$$

$$q=\frac{n(p-2)}{n-2}$$

corollary 5:

In any connected plane (p,q) graph $(p \geq 3)$ with r faces $q \geq 3r/2$ and $q \leq 3p-6$.

Proof:

case 1: Let G be a tree.

then $r=1$, $q=p-1$ and $p \geq 3$.

Hence $q \geq 3r/2$ and $q \leq 3p-6$

since $p-1 \leq 3p-6$ (as $p \geq 3$).

case 2: Let G have a cycle.

Let f_i, ($i=1$ to r) be the faces of G.

since each edge lies on the boundary of almost two faces.

$$2q \geq \sum_{i=1}^{r} (\text{number of edges in the boundary of face } f_i).$$

ie, $2q \geq 3r$ since each face is bounded by at least three edges.

ie, $q \geq 3r/2 \rightarrow (1)$.

By Euler's formula. $p-q+r=2$.

Substituting for r in (t), we get $q \geq 3/2(2+q-p)$ which on simplification gives $q \leq 3p-6$.

Definition:

A graph is called maximal planar if no line can be added to it without losing planarity. In a maximal planar graph, each face is a triangle such a graph is sometimes called a triangulated graph.

The following corollary follows directly from corollary 3 and the fact that maximal planar graph every face is a triangle.

Corollary 4: If G is a maximal planar

Corollary 5: If G is a plane connected (p, q) graph without triangles and $\underline{?}$ then $q \leq 2p-4$.

Proof: If G is a tree, then $q = p-1$.

Hence we have $p-1 = q \leq 2p-4$ ($\because$ $2 \leq p$). Now let G have a cycle.

Since G has no triangles, the boundary of face has at least four edges. Since each edge lies on at most two faces we

$$2q \geq \sum_{i=1}^{r} (\text{number of edges in the boundary of the } i\text{th face}).$$

ie, $2q \geq 4r$.

But $p-q+r = 2$ by Euler's formula.

Substituting for r in (A), we get

$2q \geq 4(2+q-p)$.

Hence $4p-8 \geq 2q$ so that $q \leq 2p-4$

Corollary 6: The graphs K_5 and $K_{3,3}$ are not planar.

Proof:

K_5 is a $(5, 10)$ graph.

For any planar graph $q \leq 3p-6$ by corollary 3.

But $q = 10$ and $p = 5$ do not satisfy

this inequality.

Hence K_5 is not planar.

$K_{3,3}$ is a $(6,9)$ bipartite graph and hence has no triangles. If such a graph planar, then by corollary 5, $q \leq 2p-4$.

But $p=6$ and $q=9$ do not satisfy this inequality.

Hence $K_{3,3}$ is not planar.

Corollary 7: Every planar graph G with $p \geq 3$ points has at least three ... of degree less than 6.

Proof:

By corollary 3, $q \leq 3p-6$

ie, $2q \leq 6p-12$

ie, $\Sigma d_i \leq 6p-12$ where d_i are the degrees of the vertices of G. ∴ G is connected $d_i \geq 1$ for every i. If at most two d_i, are less than 6,

$\Sigma d_i \geq 1+1+6+\cdots (p-2)$ times $= 6p-10$ which contradicts (1).

Hence $d_i \le 6$ for at least three values of i.

Thm 8.7.

Every planar graph G with at least 3 points is a subgraph of a triangulated graph with the same number of points

Proof.

Let G have p vertices If $p \le 3$ then G must be a subgraph of K_p which is a triangulated (maximal planar). graph. Hence let $p \ge 3$

We construct a triangulated graph G' which contains G has a subgraph as follows.

consider a plane embedding of G. If R is a face of G and v_1 and v_2 are two vertices on the boundary of R without a connecting edge we connect v_1 and v_2 with an edge lying entirely in R. This yields a new plane graph. This operation is continued until every pair of vertices on the boundary of the same face are connected by any edge. The number of vertices remain the same under these operations and hence the process terminations after some time yielding a plane triangulated graph G'. G is obviously a subgraph of G'.

This theorem is of great use in the following sense. To prove "4-colourability" for every planar graph. One common approach often used is to prove 4-colourability for maximal planar graphs (triangulations) as it is rather easier to deal with maximal planar graphs that with arbitary planar graph.

Characterization of planar graphs.

Definition:

Let $x = uv$ be an edge of a graph G. Line x is said to be subdivided when a new point w is adjoined to G and the line x is replaced by the lines uw and wv.

This process is also called an elementary subdivision of the edge x.

Two graphs are called homeomorphic (or isomorphic to within vertices of degree 2) if both can be obtained from the same graph by a sequence of subdivisions of the lines.

For example, any two cycles are homeomorphic.

Solved Problems:

1. If a (p_1, q_1) graph and a (p_2, q_2) graph are homeomorphic then $p_1 + q_2 = p_2 + q_1$

Solution:

Let the (p_1, q_1) graph G_1 and the the (p_2, q_2) graph G_2 be homeomorphic. Therefore G_1 and G_2 can be got from a (p, q) graph G by a series of elementary subdivisions (say r and s subdivisions respectively). In each elementary subdivisions (say r and s subdivisions respectively). the number of points as well as the number of edges increase by one.

Hence $p_1 = p+r$; $q_1 = q+r$; $p_2 = p+s$;
and $q_2 = q+s$

Hence $p_1 + p_2 = p+r+q+s = (p+s)+(q+r)$
$= p_2 + q_1$

The following important result known as kuratowski's theorem gives a necessary and sufficient condition for a graph to be planar.

Thm 8.8. (Kuratowski, 1930).

A graph is planar iff it has no subgraph homeomorphic to K_5 or $K_{3,3}$.

The proof of the above theorem is not given here as it is beyond the scope of this book. The graphs K_5 and $K_{3,3}$ are called Kuratowski's graphs because of their role in the above theorem. The petersen graph is not planar as it contains a subgraph homeomorphic to $K_{3,3}$. (verify).

Definition:

Let u and v be two adjacent points in a graph G. The graph obtained from G by the removal of u and v and the addition of a new point w adjacent to those points to which u or v was adjacent is called an elementary contraction of G. A graph G is contractible to a graph H if H can be obtained from G by a sequence of elementary contractions.

For example, the petersen graph given in Fig 5.6 is contractible to K5 by contracting the lines 1a, 2b, 3c, 4d, and 5c.

Thm 8.9

A graph is planar iff it does not have a subgraph contractible to K_5 or $K_{3,3}$.

Since the petersen graph is contractible to K_5, it is not planar according to the above theorem.

Definition:

Given a plane graph G, its geometric dual G^* is constructed as follows. place a vertex in each face of G (including the exterior face). For each edge x of G, draw an edge x^* joining the vertices representing the faces on both sides of x.

such that x^* crosses only the edge x. The result is always a plane graph G^* (possibly with loops and multiple edges).

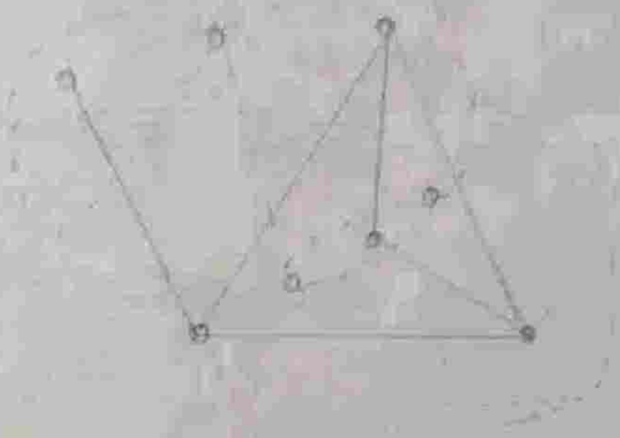


Fig. 8.4

Solved Problem:

1. Show that there is no map of five regions in the plane such that every pair of regions are adjacent.

Soln: If possible, let G be a plane map having 5 regions, such that every pair regions are adjacent.

let G^* be the geometric dual of G.
Clearly G contains five point which are mutually adjacent.

Thus K_5 is a subgraph of G^*
So that G^* is not planar,
(by Kuratowski theorem). This contradicts the fact the geometric dual is planar.

Hence the result follows.

Thickness, Crossing and outer planarity.

Definition:

The minimum number of planar subgraphs whose union is the given G is called the thickness of G and is denoted by $\theta(G)$.

If the graph G denotes an electrical circuit then the thickness of G denotes the minimum number of insultation layers needed while constructing the physical circuit.
By definition, The thickness of a

planar graph is one. The thickness of each of kuratowski's graphs is two.

Defintion:-

The crossing number of a graph G is the minimum number of pairwise intersections of its edges when G is drawn in the plane.

The crossing number of a planar graph is zero. The crossing number of each of the kuratowski's graphs is zero. The crossing number of only a few graphs have so far been determined.

Definition:

A planar graph is called outerplanar if it can be embedded in the plane so that all its vertices lie on the same face. This face is often chosen to be the exterior face.

Definition:

The graph with solid lines in fig. 8.21. is in fact outerplanar since it can be given an outer planer embedding. It is obvious that a graph G is outerplanar iff each of its blocks is outerplanar.

Definition:

An outerplanar graph is called maximal outerplanar if no line can be added without losing outerplanarity. obviously, every maximal outerplanar graph is a triangulation of a polygon while every maximal plane graph is a triangulation of the sphere.

Definition:

The genus of a graph G is defined to be the minimum number of handles to be attached to a sphere so that G can be drawn on the resulting surface without intersecting lines.

Every planar graph has genus 0.

K_5, K_6, K_7, $K_{3,3}$ and $K_{4,4}$ each have genus 1.

Colourability:

chromatic number and chromatic index:

Definition:

An assignment of colours to the vertices of a graph so that no two adjacent vertices get the same colour is called a colouring of the graph. For each colour, the set of all points which get that colour is independent and is called a colour class.

A graph colouring a graph G using at most n colours is called an n colouring. The chromatic number $\chi(G)$ of a graph G is the minimum number of colours needed to colour G. A graph G is called

G chromatic n-colourable if $\chi(G) \leq n$.

Example:

Graph	K_p	$K_p - x$	$\bar{K_p}$	$K_{m,n}$	C_{2n}	C_{2n+1}
Chromatic Number	p	$p-1$	1	2	2	3

When T is a tree with atleast two points $\chi(T) = 2$.

A wheel has chromatic number 3 or 4 according as it has an odd or even number of points.

Definition:

Each n-colouring of G partitions $V(G)$ into n independent sets called colour classes such a partitoning induced by a $\chi(G)$ colouring of G is called a chromatic partitoning.

In other words, a partition of $V(G)$ into the smallest possible number of independent sets is called a chromatic partitioning of G.

Example:

{1, 4, 8} {3, 6, 7} {2, 5} is a chromatic partitioning of a graph in Fig 9.1 which has chromatic number 3.

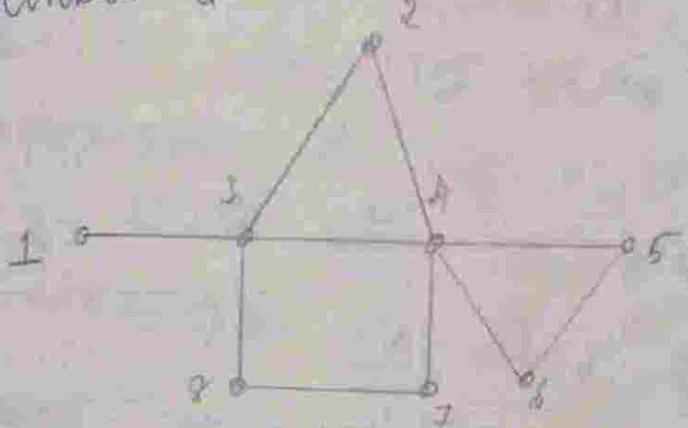


Fig. 9.1

Thm 9.1 The Following statements are equivalent for any graph G.

i) G is 2- colourable.

(ii) G is bipartic

(iii) Every cycle G has even length.

Proof:

(i) => (ii). G is 2 colourable. Hence V(G) can be partitioned into two colour classes. These colour classes are independent sets and hence

Form a bipartition of G. Hence G is bipartite.

(ii) ⇒ (i). G is bipartite. Hence $V(G)$ can be partitioned into two sets V_1 and V_2 such that V_1 and V_2 are independent sets. A 2-colouring of G can be obtained by colouring all the points of V_2 blue. Hence G is 2-colourable.

(ii) ⇔ (iii) follows from thm 4.7.

<u>Remark:</u>

G is bipartite does not imply $\chi(G) = 2$. For example $\overline{K}_2$, which is bipartite has chromatic number 1. However if G has an edge and is bipartite then $\chi(G) = 2$.

<u>Definition:</u>

<u>Critical.</u>

A graph G is called critical if $\chi(H) < \chi(G)$ for every proper

Subgraph H of G. A-chromatic graph that is critical is called K-critical. It is obvious that every K-chromatic graph has a K-critical subgraph.

Thm 9.2

If G is K-critical, then $\delta(G) \geq K-1$.

Proof:

Since G is K-critical, for any vertex v of G, $\chi(G-v) = K-1$. If $\deg v < K-1$, then a $(K-1)$-colouring of $G-v$ can be extended to a $K-1$ colouring of G by assigning to v, a colour which is assigned to none of its neighbours in G. Hence $\deg v \geq K-1$. So that $\delta(G) \geq K-1$.

Corollary: 1. Every K-chromatic graph and has at least K vertices of degree at least $K-1$.

Proof://

Proof:

Let G be a k-chromatic graph and H be a k-critical subgraph of G. (By thm 9.2) $\delta(H) \geq k-1$. Also since $\chi(H) = k$, H has at least k vertices. Hence H has at least k vertices of degree at least $k-1$ since H is a subgraph of G. The result follows.

Corollary 2. For any graph G, $\chi \leq \Delta + 1$.

Proof:

Let G have chromatic number χ. Let H be a χ-critical subgraph of G. (By thm 9.2) $\delta H \geq \chi - 1$. Hence $\chi \leq \delta(H) + 1$. Since $\delta(H) \leq \Delta(G)$. This implies that $\chi \leq \Delta(G) + 1$.

Thm 9.3

For any graph G, $\chi(G) \leq 1 + \max \delta(G')$ where the maximum is taken over all induced subgraphs G' of G.

Proof:

The theorem is obvious for totally disconnected graphs. Now let G be an arbitrary n-chromatic graph $n \geq 2$. Let H be any smallest induced subgraph of G such that $\chi(H) = n$.

Hence $\chi(H-v) = n-1$ for every point v of H.

If $\deg_H v < n-1$, then a $(n-1)$. colouring of $H-v$ can be extended to a $n-1$ colouring of H by assigning to v, a colour which is assigned to none of its neighbours in H.

Hence $\deg_H v \geq n-1$. Since v was arbitrary vertex of H, this implies that $\delta(H) \geq n-1 = \chi(G)-1$.

Hence $\chi(G) \leq 1 + \delta(H) \leq 1 + \max \delta(H')$ where the maximum is taken over the set A of induced subgraphs H' of H. Hence $\chi(G) \leq 1 + \max \delta(G')$, where the maximum is taken over the set B of induced subgraphs G' of G.

Definition:

If $\chi(G) = n$ and every n-colouring of G induces the same partition on $V(G)$ then G is called uniquely n-colourable or uniquely colourable.

K_3 and $K_4 - x$ are uniquely 3-colourable. K_n is uniquely n-colourable. $K_n - x$ is uniquely $(n-1)$ colourable. Any connected bipartite graph is uniquely 2-colourable.

Thm 9.4

If G is uniquely n-colourable, then $\delta(G) \geq n-1$.

Proof:

Let v be any point of G. In any n-colouring, v must be adjacent with at least one point of every colour different from the assigned to v. Otherwise, by recolouring v with a colour which none of its neighbours is having, a different n-colouring can be achieved. Hence degree of v is at least $n-1$ so that $\delta(G) \geq n-1$.

Thm 9.5

Let G be a uniquely n-colourable graph. Then in any n-colouring of G, the subgraph induced by the union of any two colour classes is connected.

Proof:

If possible, let C_1 and C_2 be two classes in a n-colouring of G and that the subgraph induced by $C_1 \cup C_2$ is disconnected. Let H be a component of the subgraph induced by $C_1 \cup C_2$. Obviously, no point of H is adjacent a point in $V(G) - V(H)$ that is coloured C_1 or C_2. Hence interchanging the colours of the points in H and retaining the original colours for all other points we get a different n-colouring for G. This gives a contradiction.

Note: This type of interchange of colours in a subgraph is used often in study of colourings.

Thm 9.6

Every uniquely n- colourable graph is (n-1) connected.

Proof:

Let G be a uniquely n- colourable graph consider an n- colouring G. If possible let G be not (n-1) connected. Hence there exists a set S at most n-2 points such that G-S is either trivial or disconnected. If G - S trivial, then G has at most n-1 points so that G is not uniquely n- colourable. Hence G-S has at least two components. In the considered n- colouring. There are at least two colours say c1 and c2 that are not assigned to any point of G.

If every point in a component

of $G-S$ has colour different from c_1 and c_2 then by assigning colour c_1 to a point of this component, we get a different n- colouring of G. otherwise, by interchanging the colours c_1 and c_2 in component of $G-S$, a different n- colouring of G is obtained. In any colour G is not uniquely n- colourable, giving a contradiction. Hence G is $(n-1)$ connected.

corollary:

In any n- colouring of a uniquely n- colourable graph G, subgraph induced by the union of any k colour classes $2 \leq k \leq n$, is $(k-1)$ connected.

Proof:

If the subgraph H induced by the union of any k colour class $2 \leq k \leq n$, had different k- colourings, then these k- colourings will induced different n- colourings for G giving a contraction. Hence H is uniquely the colourable. Hence by thm 9.6. H is (k-1) connected.

Definition:

An assignment of colours to the edges of a graph G so that the two adjacent edges get the same colour is called an edge colouring or line colouring of G.

An edge colouring of G using n colours is called a n-edge colouring. (or n-line colouring).

The edge chromatic number (also called line chromatic number or chromatic index).

$x'G$ is the minimum number of colours needed to edge colour G.

A graph G is called the n-edge colourable if $x'(G) \leq n$.

Tight bounds on the line chromatic number were found by Vizing.

Thm 9.7 (Vizing 1964)

For any graph G, the edge chromatic number is either Δ or $\Delta+1$

The proof of the thm is beautiful but lengthy and hence is not included.

For K_4, $\Delta = 3$ and $\chi' = 3$.

For C_3, $\Delta = 2$ and $\chi' = 3$.

Thm 9.8

$\chi'(K_n) = n$ if n is odd $(n \neq 1)$ and

$\chi'(K_n) = n-1$ if n is even.

Proof:

If $n = 2$, the result is obvious. Hence let $n > 2$. Let n be odd. Now the edges of K_n can be n-coloured as follows. Place the vertices of K_n in the form of a regular n-gon. Colour the edges around the boundary using a different colour for each edge.

Let x be any one of the remaining edges. x divides the boundary into two segments, one say B_1 containing an odd number of edges and other containing an even number of edges. colour x with the same colour as the edge that occurs in the middle of B_1. Note that these two edges are parallel. The result is a n-edge colouring of K_n since any two edges having the same colour are parallel and hence are not adjacent.

Hence $\chi'(K_n) \leq n$. $\rightarrow$ (1).

Since K_n has n points and n is odd. It can have at most $\frac{1}{2}(n-1)$ mutually independent edges. Hence each colour class can have at most $\frac{1}{2}(n-1)$ edges, so that the number of colour classes is at least.

$$\binom{n}{2} \Big/ \tfrac{1}{2}(n-1) = n \text{ so that } \chi'(K_n) \geq n$$

$\rightarrow$ (2).

(1) and (2) together imply $\chi'(K_n) = n$.

Let $(n \geq 4)$ be even. Let K_n have vertices $v_1, v_2, \ldots, v_n$. Colour the edges of the subgraph K_{n-1} induced by the first $n-1$ points using the method described above. In this colouring, at each vertex, one colour (the colour assigned to the edge opposite to this vertex on the boundary) will be missing. Also, these missing colours are all different. This edge colouring of K_{n-1} can be extended to an edge colouring of K_n by assigning the colour that is missing at v_i to edge $v_i v_n$ for every $i, i < n$.

Hence $\chi'(K_n) \leq n-1$. Also

$\chi'(K_n) \geq \Delta(K_n) = n-1$.

Hence $\chi'(K_n) = n-1$.

The Five colour theorem.

Heawood (1890) showed that one can always colour the vertices of a planar graph with at most five colours. This is known as the Five colour theorem.

Thm 9.1

Every planar graph is 5-colourable.

Proof:

We will prove the theorem by induction on the number p of points any planar graph having $p \leq 5$ points, the result is obvious since the graph p-colourable.

Now let us assume that all planar graphs with p points is 5-colourables some $p \geq 5$. Let G be a planar graph with p+1 points. Then G has a vertex v of degree 5 or less. (corollary 7 to theorem). By induction hypothesis the plane graph

$G-v$ is 5-colourable. consider a 5-colouring of $G-v$ which $c_i, 1 \le i \le 5$ are the colours used. If some colour, say c_j is not used in colouring vertices adjacent to v, then by assigning the colour c_j to v the 5 colouring $G-v$ can be extended to a 5-colouring of G.

Hence we have to consider only the case in which $\deg v = 5$ and all the colours are used for colouring the vertices of G adjacent to v.

let v_1, v_2, v_3, v_4, v_5 be the vertices adjacent to v coloured c_1, c_2, c_3, c_4 and respectively.

Let G_{13} denote the subgraph of $G-v$ induced by those vertices coloured or c_3. if v_1 and v_3 belong to different components of G_{13}, then a 5-colouring $G-v$ can be obtained interchanging the colours of vertices in the components of G_{13} containing v_1.

(since no point of this component is adjacent to a point with colour c_1 or c_3 outside this component, this interchange of colours result in a colouring of $(G-v)$. In this 5-colouring no vertex adjacent to v is coloured c_1 and hence by colouring v with c_1 a 5-colouring of G is obtained.

If v_1 and v_3 are in the same component of G_{13}, then in G there exists a v_1-v_3 path all of whose points are coloured c_1 or c_3. Hence there is no v_2-v_4

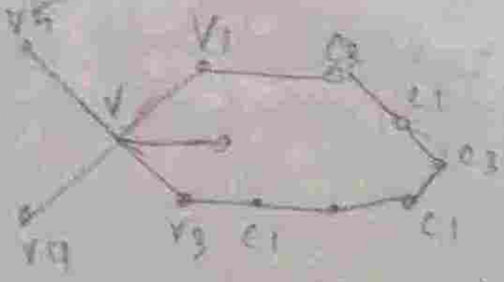


Fig (9-2)

Hence if G_{24} denotes the subgraph of $G-v$ induced by the points coloured c_2 or c_4, then v_2 and v_4 belong to different components of G_{24}. Hence if we interchange the colours of the points in the component of G_{24} containing v_2, a new 5-

colouring $G-v$ results and in this no point adjacent to v is coloured c_2. Hence by assigning colour c_2 to v, we can get a 5-colouring of G. This completes the induction and the proof.

Five colour Problem

The Four colour conjecture states that any map on a plane or on the surface of a sphere can be coloured with only four colours so that no two adjacent countries have the same colour. Each country must consist of a single connected region and adjacent countries are those having a boundary line (not merely a single point) in common. The problem of deciding whether the four colour conjecture is true or false is called the four colour problem.

As seen in section 7.2 a plane graph (geometric dual) can be associated with each map. colouring the countries of the map is equivalent to colouring the vertices of its geometric dual. In this set up, the four colour conjecture states that "Every planar graph is 4- colourable".

The Number 4 cannot be reduced further as there are maps that require at least four colours. The map in Fig 9.3 is one such map.

Fig 9.3

There are several problems in graph theory that are equivalent to the four colour problem. One of these in the case $n=5$. of the following conjecture.

Hadwiger's conjecture. Every connected n- chromatic graph is contradiction to K_n.

Another equivalent of the four colour conjecture (4CC) is given by following.

Thm 9.10

The ACC is true iff every bridgeless cubic plane map colourable (A map is said to be m colourable if its regions can be coloured m or fewer colours so that adjacent regions have different colours).

proof:

ACC holds <=> every plane map is a 4- colourable (obvious) <=> the bridgeless plane map is 4- colourable.

(since identification of the end vertices of a bridge affects neither the number regions nor the adjacency among regions in the map).

obviously every bridgeless plane map is 4- colourable => every cubic bridge plane map is 4- colourable.

We now proceed to prove the converse of (2).

Assume that every cubic bridgeless plane map is 4 colourable.

Let G be the bridgeless plane map.

Since G is bridgeless, it has no endpoints. If G contains a vertex of degree 2 adjacent to the vertices u and w replace it by $K_4 - X$ for that its vertices of degree 2 are adjacent to u and w.

If G contains a vertex v of degree $n \geq 4$ adjacent to the vertices $v_1, \ldots$ arranged cyclically about v replace v by a cycle $u_1, u_2, \ldots u_n, u_1$ of length and join u_i and v_i by an edge for every i. In both cases, each new point adjacent has degree 3 and the adjacency between the original regions of G are present. Repeat this process for every vertex v of G with deg $v \neq 3$. let G' denotes the resulting cubic bridgeless plane map. By hypothesis there is a 4-colouring of the regions of G'. For each vertex of G with deg $v \neq 3$, if we identify the newly introduced vertices

Unit - 5

Theorem 9.14

If G is a graph with k components $G_1, G_2, \dots G_k$

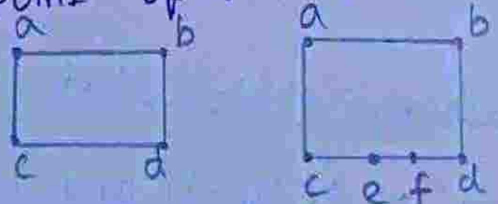
then $f(G,\lambda) = \prod_{i=1}^{k} f(G_i, \lambda)$

Number of ways of colouring G_i with λ colours is $f(G,\lambda)$. since the choice of λ-colorouing for $G_1, G_2, \dots G_k$ can be combined to give a λ-colouring G. $f(G,\lambda) = \prod_{i=1}^{k} f(G_i, \lambda)$

Definition

Let u and v be two non adjacent points in a graph G. The graph obtained from G by the removal of u and v and the addition of a new points adjacent to those point to which u or v was adjacent is called a Elementary homomorphism

In other words identification of two nonadjacent point of G is called an elementary homomorphism.



Theorem 9.15

If u and v are non adjacent point in a graph G and the donates the elementary homomorphism of G which identifies u and v. the $f(G, \lambda) = f(G+uv, \lambda) + f(hG, \lambda)$ where $G+uv$ denotes the graph obtained from G by adding the line uv.

proof:

$f(G, \lambda)$ = number of colouring of G from λ colours

= number of colouring G from λ colours in (which u and v get different colours) +

number of colouring of G from λ colours in which u and v from get the same colour.

= (number of colouring of $G+uv$ from λ colours)

+ (number of colouring of hG from λ colours)

$$f(G, \lambda) = f(G+uv, \lambda) + f(hG, \lambda)$$

corollary

(i) For any graph G, $f(G, \lambda)$ is a polynomial in λ

(ii) $f(G, \lambda)$ has degree $|V(G)|$

(iii) The constant term in $f(G, \lambda)$ is 0

The above theorem states that $f(G, \lambda)$ can be written as the sum of $f(G_1, \lambda)$ and $f(G_2, \lambda)$ where G_1 has the same number of points as G with one or more edge and G_2 has one point less than G

Doing this process repeatedly $f(G, \lambda)$ can be written as $\sum f(G_i, \lambda)$ where each G_i is a complete graph as $\max |V(G_i)| = |V(G)|$

Since $f(K_n, \lambda)$ is a polynomial of degree n, it follows that $f(G, \lambda)$ is a polynomial of degree $|V(G)|$

Since $f(K_n, \lambda)$ has constant term 0, the constant term in $\sum f(G_i, \lambda)$ is 0 so that (iii) holds.

Theorem 9.11

(Four colour Theorem). Every planar graph is 4-colourable.

A computer-free proof of the above theorem is still to be found.

Definition:-

Let S_n be an orientable surface of genus n. (S_n is topologically equivalent to a sphere with n handles and so is the ordinary sphere).

Let S be an orientable surface. The chromatic number of S, denoted by $\chi(S)$, is defined to be $\chi(S) = \max\{\chi(G) / G$ is a graph embeddable on $S\}$.

Theorem 9.12

(Heawood map colouring Theorem). For every positive integer n, $\chi(S_n) = \left[\frac{7+\sqrt{(1+8n)}}{2}\right]$

The final proof of this Theorem was given by Ringel and youngs. Note that four colour theorem is the extension of the above Theorem to the case $n=0$

CHROMATIC POLYNOMIALS

Birkhoff (1912) introduced chromatic polynomials as a possible means of attacking the four colour conjecture. This concept considers the number of ways of colouring a graph with given number of colours.

Let G be a labled graph. A colouring of G from λ colours is a colouring of G which uses λ or fewer colours. Two colourings of G from λ colours will be considered different if at least one of the labled points is assigned different colours. Let $f(G;\lambda)$ denote the number of different colourings of G from λ colours.

For example,

$$f(K_1;\lambda)=\lambda \text{ and } f(\overline{K_2},\lambda)=\lambda^2$$

Theorem 9.13

$$f(K_n,\lambda)=\lambda(\lambda-1)\cdots(\lambda-n+1)$$

Proof:

The first vertex in K_n can be coloured in λ different ways (as there are λ Theorm).

For each colouring of the first vertex,

The second vertex can be Coloured in $\lambda-1$ ways (as there are $\lambda-1$ colours remaining).

For each colaring of the first two vertices, the third can be coloured $\lambda-2$ ways and so on.

Hence $f(K_n, \lambda) = \lambda(\lambda-1)\cdots\cdots(\lambda-n+1)$

Remark:-

$f(\overline{K_n}, \lambda) = \lambda^n$, since each of the n points of K_n may be coloured independently in λ ways.

Theorem 10.10:

Each vertex of a disconnected tournament D with at least p points $(p \geq 3)$ is contained in a directed cycle of length k, for every k, $3 \leq k \leq p$.

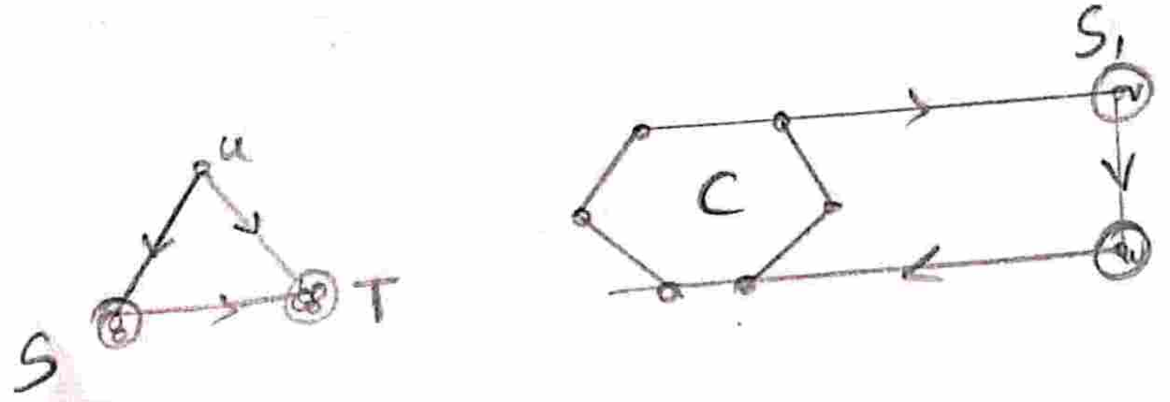
~~Pf~~:

Proof: Let us denote.

$N^+(u) = \{w/(u,w)$ is an arc$\}$ and

$N^-(u) = \{w/(w,u)$ is an arc$\}$

Let u be any point of D. Let $S = N^+(u)$ and $T = N^-(u)$. clearly $S \cap T = \phi$ and $V(D) - \{u\} = S \cup T$. Since D is disconnected, neither S nor T is empty Also T must be reachable from S. Hence there exists an arc from a point v . S to a point w of T.



Hence u lies on the directed 3-cycle $uvwu$,

The theorem is now proved by induction on k. Suppose that u is in a directed cycle of all lengths between 3 and n where $n < p$. We shall that u is in a directed $(n+1)$-cycle.

Let $C = v_0, v_1, \ldots v_n$ be a directed n-cycle in which $v_0 = v_n = u$. if where is a point w_1 in $V(D) - V(C)$ such that $v_i w_1$ and $w_1 v_j$ are arcs for some i and j, $1 \leq i, j \leq n$, then there are adjacent points v_k and v_{k+1} in c such that both $v_k w_1$ and $w_1 v_{k+1}$ are arcs of D. In this case u is in the directed $(n+1)$-cycle $v_0, v_1 \ldots v_k w_1 v_{k+1}$. otherwise, Let $S_1 = \{w / w \in V(D) - V(C)$ and all arcs between c and w are directed towards $w\}$ and $T_1 = \{w / w \in V(D) - V(C)$ and all arcs between c and w are directed from $w\}$.

Since D is disconnected, T_1 must be reachable from S_1. Hence there exists $v \in S_1$ and $w \in T_1$ such that vw is an arc of D. Hence u is in the directed $(n+1)$ - cycle $v_0 v w v_2 v_3 \ldots v_n$.

This completes the induction and the proof.

Corollary:-

Every tournament is either diconnected or can be transformed into a diconnected one by the reorientation of just one arc.

proof:-

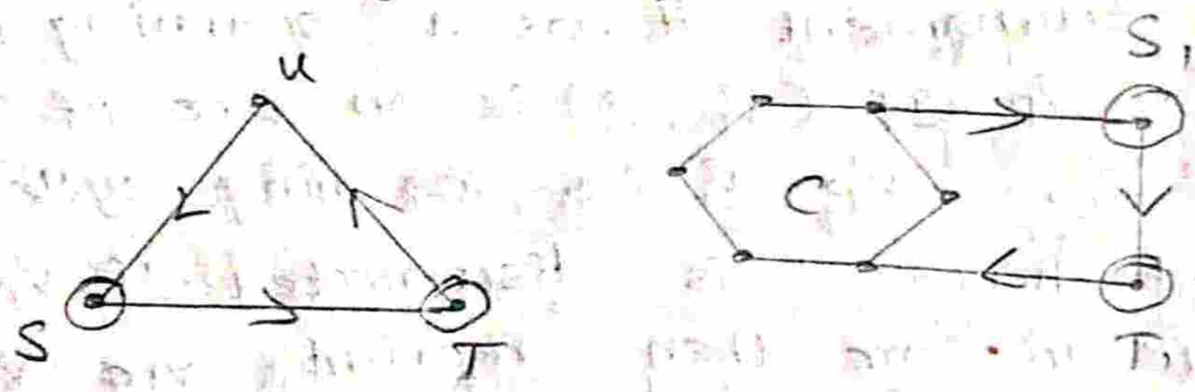
Let T be any tournament. If it disconnected, there is nothing to prove. Hence let T be not disconnected. Since T is a tournament it has a spanning path $v_1, \ldots, v_p$. If (v_p, v_1) is an arc of T then $v_1, \ldots v_p, v_1$ is spanning cycle in T and hence T is disconnected. If $v_p v_1$ is not an arc then reorient arc (v_1, v_p) as an arc from v_p to v_1 getting a tournament T'. Now, $v_1, \ldots v_p, v_1$ is a spanning cycle in T' so that T' is diconnected.

Theorem 10.10

Each vertex of a diconnected tournament D with at least p points $(p \geq 3)$ is contained in a directed cycle of length k, for every k, $3 \leq k \leq p$.

proof:- $N^+(u) = \{w / (u, w)$ is an arc$\}$ and $N^-(u) = \{w / (w, u)$ is an arc$\}$

Let u be any point of D. Let $S = N^+(u)$ and $T = N^- \{u\}$. clearly $S \cap T = \phi$ and $V(D) - \{u\} = S \cup T$. Since D is diconnected, neither S nor T is empty. Also T must be reachable from S. Hence there exists an arc from a point v of S to a point w of T (Fig 10.8 (a))



Hence u lies on the directed 3-cycle $uvwu$.

The theorem is now proved by induction on k. Suppose that u is in a directed cycle of all lengths between 3 and n where $n < p$. We shall show that u is in a directed $(n+1)$-cycle.

Let $C = v_0, v_1, \ldots, v_n$ be a directed n-cycle in which $v_0 = v_n = u$. If there is a point w_1 in $V(D) - V(C)$ such that $v_i w_1$ and $w_1 v_j$ are arcs for some i and j, $1 \leq i, j \leq n$, then there are adjacent points v_k and v_{k+1} in C such that both $v_k w_1$ and $w_1 v_{k+1}$ are arcs of D. (As in the proof of Redei's Theorem). In this case u is in the directed $(n+1)$-cycle $v_0, v_1, \ldots, v_k w_1 v_{k+1} \ldots v_n$. Otherwise, let $S_1 = \{w / w \in V(D) - V(C)\}$ and all arcs between C and w are directed towards $w\}$ and $T_1 = \{w / w \in V(D) - V(C)\}$ and all arcs between C and in arc directed from $w\}$.

Since D is diconnected, T_1 must be reachable from S_1. Hence there exists $v \in S_1$ and $w \in T_1$ such that vw is an arc of D (Fig 10.8(b)). Hence u is in the directed $(n+1)$-cycle $v_0 v w v_2 v_3 \ldots v_n$.

This completes the induction and the proof.

Graph Theory

Corollary:-

Every tournament is either diconnected or can be transformed into a diconnected one by the reorientation of just one arc.

Proof:-

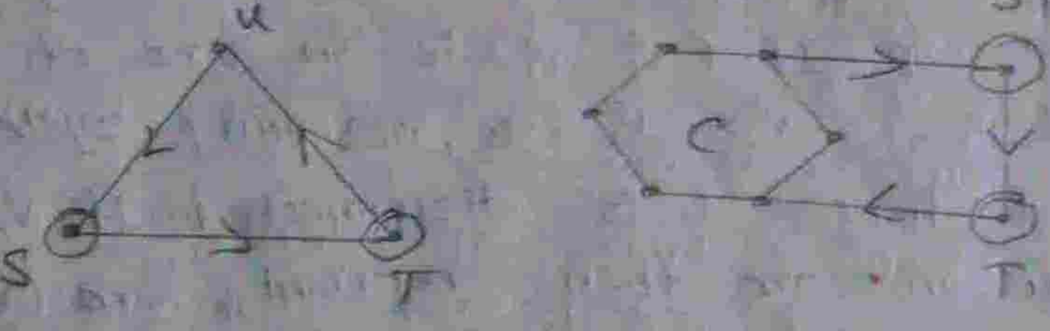
Let T be any tournament. If it disconnected, there is nothing to prove. Hence let T be not disconnected. Since T is a tournament it has a spanning path $v_1, \ldots, v_p$. If (v_p, v_1) is an arc of T then $v_1, \ldots v_p, v_1$ is spanning cycle in T and hence T is disconnected. If $v_p v_1$ is not an arc then reorient arc (v_1, v_p) as an arc from v_p to v_1 getting a tournament T'. Now, $v_1, \ldots v_p, v_1$ is a spanning cycle in T' so that T' is diconnected.

Theorem 10.10

Each vertex of a diconnected tournament D with at least p points ($p \geq 3$) is contained in a directed cycle of length k, for every k, $3 \leq k \leq p$.

proof:- $N^+(u) = \{w / (u,w) \text{ is an arc}\}$ and $N^-(u) = \{w / (w,u) \text{ is an arc}\}$

Let u be any point of D. Let $S = N^+(u)$ and $T = N^-\{u\}$. clearly $S \cap T = \phi$ and $V(D) - \{u\} = S \cup T$. Since D is diconnected neither S nor T is empty. Also T must be reachable from S. Hence there exists an arc from a point v of S to a point w of T. (Fig 10.8 (a))



Hence u lies on the directed 3-cycle uvwu.

The theorem is now proved by induction on k. Suppose that u is in a directed cycle of all lengths between 3 and n where $n < p$. We shall show that u is in a directed $(n+1)$-cycle.

Let $C = v_0, v_1, \ldots, v_n$ be a directed n-cycle in which $v_0 = v_n = u$. If there is a point w_1 in $V(D) - V(C)$ such that $v_i w_1$ and $w_1 v_j$ are arcs for some i and j, $1 \leq i, j \leq n$, then there are adjacent points v_k and v_{k+1} in C such that both $v_k w_1$ and $w_1 v_{k+1}$ are arcs of D. (As in the proof of Redei's Theorem). In this case u is in the directed $(n+1)$-cycle $v_0, v_1, \ldots, v_k w_1 v_{k+1} \ldots v_n$. Otherwise, let $S_1 = \{w \mid w \in V(D) - V(C)\}$ and all arcs between C and w are directed towards $w\}$ and $T_1 = \{w \mid w \in V(D) - V(C)\}$ and all arcs between C and w are directed from $w\}$.

Since D is diconnected, T_1 must be reachable from S_1. Hence there exists $v \in S_1$ and $w \in T_1$ such that vw is an arc of D (Fig 10.8(b)). Hence u is in the directed $(n+1)$-cycle $v_0 v w v_2 v_3 \ldots v_n$.

This completes the induction and the proof.

Digraphs and Matrices:-

Definition:-

Let D be a digraph with p vertices. The adjacency matrix or dominance matrix $A(D)$ of D is a $p \times p$ matrix (a_{ij}) with

$$a_{ij} = \begin{cases} 1 & \text{if } v_i v_j \text{ is an arc of } D \\ 0 & \text{otherwise.} \end{cases}$$

The digraphs in fig 10.1, 10.2, and 10.5 have adjacency matrices.

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \qquad \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{array}{c} \\ A \\ B \\ C \\ D \\ E \\ F \\ G \end{array} \begin{array}{c} \begin{array}{ccccccc} A & B & C & D & E & F & G \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

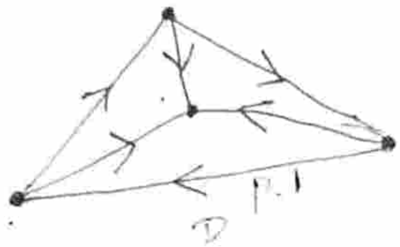
respectively. The Sum of th i^{th} row entires of A (D) gives $d^+(v_i)$ and the Sum of the i^{th} Column entires of A(D) gives $d^-(v_i)$ for every i.

The powers of A (D) give the number of walks from one point to another as shown in the following theorem.

tournament

Definition:

A digraph D is called a tournament if for every pair of points u and v in D there is exactly one arc between u and v. The score of a point in a tournament is its outdegree.



In a tournament with p points the sum of outdegree and indegree of point is p-1 and hence from the score of a point, its indegree can also be found out.

Remark:

The score of the points of a tournament written in non-increase order is called its score sequence. In a tournament if (v, u) is an arc then v is said to dominate u

Ex: There is only one tournament on two points.

The two tournament with three points and four tournament on 4 points

3 points

4 points

The first tournament with three points is called a cycle triple and the second is called a transitive triple.

Definition

Let D be a graph with p vertices. The reachability matrix $R = (a_{ij})$ is the $p \times p$ matrix with $a_{ij} = 1$ if v_j is reachable from v_i and 0 otherwise. We assume that each vertex is reachable from itself.

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ reachability matrix}$$

The distance matrix is the $p \times p$ matrix whose $(i,j)^{th}$ entry gives the distance from the point v_i to the point v_j and is infinity if there is no path from v_i to v_j.

$$\begin{pmatrix} 0 & 1 & 2 & 3 \\ 2 & 0 & 1 & 2 \\ 1 & 2 & 0 & 1 \\ \infty & \infty & \infty & 0 \end{pmatrix} \text{ distance matrix}$$

The detour matrix is the $p \times p$ matrix whose $(i,j)^{th}$ entry is the length of any longer $v_i - v_j$ path and is infinity if there is no such path.

$$\begin{pmatrix} 0 & 1 & 2 & 3 \\ 2 & 0 & 1 & 2 \\ 1 & 2 & 0 & 1 \\ \infty & \infty & \infty & 0 \end{pmatrix} \text{ detour matrix.}$$

Theorem : 10.5

The $(i,j)^{th}$ entry A^n is the number of walks length n from v_i to v_j.

Proof :-

We will prove the theorem using induction on n. From the definition of adjancency matrix, we see that the theorem holds for $n=1$. Now assume that the theorem holds for $n-1$. Let $A^{n-1} = (b_{ij}) \rightarrow (1)$

Hence b_{ij} = number of $v_i - v_j$ walks of length $n-1$

Now, $A^n = A^{n-1} A$.

$\therefore (i,j)^{th}$ entry of $A^n = b_{i1}a_{1j} + b_{i2}a_{2j} + \dots + b_{ip}a_{pj} \rightarrow (2)$

By the definition of $A(D)$,

a_{kj} = number of $v_k - v_j$ walks of length 1. Hence by (1) for every k, $1 \leq k \leq p$, $b_{ik}a_{kj}$ = number of $v_i - v_j$ walks of length 2 whose last arc is $v_k v_j$. Since any $v_i - v_j$ walk has one among $v_1 v_j, v_2 v_j, \dots, v_p v_j$ as the last arc, the right hand side of (2) gives the number of $v_i - v_j$ walks of length n.

Hence the $(i,j)^{th}$ entry of A^n is the number of $v_i - v_j$ walks of length n.

This completes the induction and the proof.

Thm : 9.17

A graph G with n points $f(G, \lambda) = \lambda(\lambda-1)^{n-1}$ is a tree.

have the chromatic polynomial $\lambda(\lambda-1)^5$. But they are not isomorphic.

In the following solved problems we give some more properties of chromatic polynomials

Solved problems.

Pbm: 1 Prove that the coefficients of $f(G, \lambda)$ alternate in sign.

Soln

we prove the result by induction on the number of lines q. when $q=0$, $f(G, \lambda) = \lambda^p$ where p is the number of points of G. In this case then polynomial has just one non-zero coeffient and hence the result is trivially true.

Now assume that the result is true for all graphs with less than q lines. let G be a (p, q) graph with $q > 0$.

Let $e = uv$ be an edge of G.

Let $G_1 = G - uv$. Clearly u and v are nonadjacent in G_1.

Hence $f(G_1, \lambda) = f(G_1 + uv, \lambda) + f(h G_1, \lambda)$ (by Thm 9.15)

$= f(G, \lambda) + f(h G_1, \lambda)$

Hence $f(G, \lambda) = f(G_1, \lambda) - f(h G_1, \lambda) \rightarrow (1)$

Now G_1 is a $(p, q-1)$ graph and hG_1 is a $(p-1, q_1)$ graph where $q_1 < q$.

Hence by induction hypothesis

$f(G_1, \lambda) = \lambda^p - \alpha_1 \lambda^{p-1} + \alpha_2 \lambda^{p-2} - + \cdots + (-1)^{p-1} \alpha_{p-1} \lambda$

and $f(h G_1, \lambda) = \lambda^{p-1} - \beta_1 \lambda^{p-2} + \cdots + (-1)^{p-1} \beta_{p-2} \lambda$

where α_i and β_j are non-negative integers

Hence by (1)

$f(G, \lambda) = \lambda^p - (\alpha_1 + 1) \lambda^{p-1} + (\alpha_2 + \beta_1) \lambda^{p-2} - \cdots + (-1)^{p-1} (\alpha_{p-1} + \beta_{p-2}) \lambda$

This is a polynomial in which the coefficient alternate in sign

This completes the induction and the proof

Corollary:

A tournament is strong iff it has a spanning cycle.

Proof:

If the tournament D on p points is strong then by the above theorem.

it has a cycle of length p

Hence D has a spanning cycle.

Conversely, if a digraph has a spanning cycle, every pair of points are mutually reachable and hence the digraph is disconnected.

Stockmeyer in 1977 constructed a remarkable family $\{A_n / n > 0\}$ of tournaments in his attempts to disprove Ulam conjecture for diagraphs. Let $p = 2^n, n > 0$. Then A_n has vertex and set $v_1, v_2, \ldots, v_p$ and arc set $\{(v_i, v_j) \mid \text{odd}(j-i) \equiv 1 \pmod 4\}$ where for any nonzero integer k, odd(k) is the odd integer obtained on dividing k by the appropriate power of 2. Then odd$(-6) = -3$ and odd$(8) \equiv$

1. The adjacency matrices of A_2 and A_3 are respectively

$$\begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \text{ and } \left(\begin{array}{cccc|cccc} 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ \hline 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ \hline 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \end{array}\right)$$

(These can be remembered easily. Fill up the first row using the definition or from memory. First column can now be filled since A_n is a tournament. The remaining entries can now be filled remembering on four points in Fig. 10.5 as A_2

Some of the facinating properties of the tournaments A_n are given in the following theorem.

Theorem 10.3

The edges of a connected graph $G=(V, E)$ can be Oriented so that the resulting diagraph is strongly connected iff every edge of G is contained in at least one cycle.

proof:-

Suppose the edges of G can be oriented that the resulting diagraph becomes strongly connected.

If possible, let $c = vw$ be an edge of G not lying on any cycle. Now as soon as e is oriented, one of the vertices u and w becomes non-reachable from the other. Hence an orientation of the required types is not possible giving constradiction. Hence every edge of G lies on a cycle.

Let $S = v_1 v_2 \dots v_n, v_1$ be a cycle in G. Orient the edges of S so that S becomes a directed cycle and hence becomes a strongly connected Subdiagraph. If $V = (V_1 \ldots V_n)$ then we are thro otherwise, let w be a vertex of G was in S such that w is adjacent to a vertex V_1 of S, Let $C = V_1 W$. By hypothesis e lies on some cyc we choose a direction of c and give the orientation determined by the direction to the edges of C which are not already oriented. The resulting enlarged oriented graph is also strongly connected as it can be give then S by a sequence of additions of simple directed pat (for example if $v \in S$ and u is a point on a simple directed $V_n - V_1$ path p added to S the is the enlarged oriented graph the $u-u_1$ subpath of P followed by the $V_1 - V$ subpath of S give a directed $u-v$ path. Also the $V-V_1$ subpath of S followed by the $V_1 - u$ subpath p give a directed $v-u$ path. This type of argument can be repeated for each addition of simple, directed paths) (see fig 10.3)

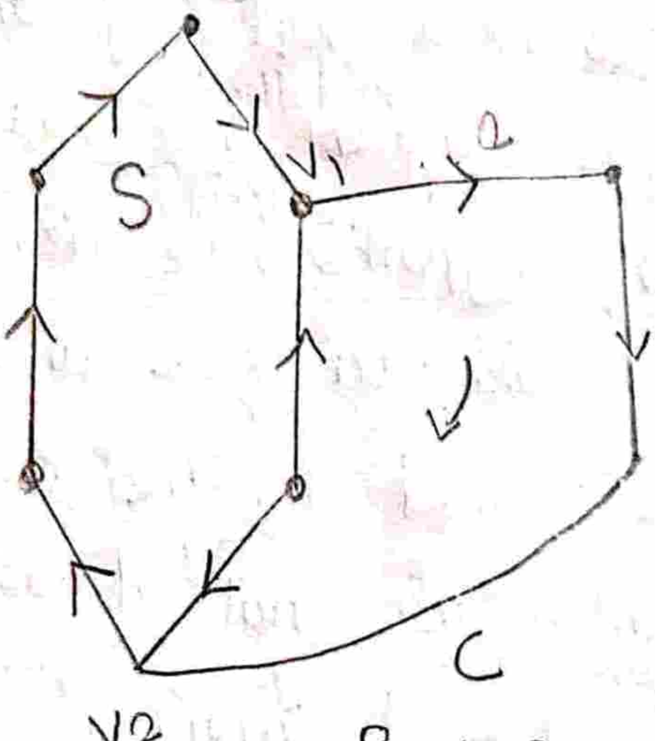


fig 10.3.

This process can be repeated till we get a strongly connected oriented spanning subgraph of G. The remaining edges can now be oriented in any way. the resulting oriented graph is strongly connected. The complete the proof.

There are three different kinds of components of a diagraph.

Theorem: 10.4

A weak digraph D is Eulerian iff every point of D has equal indegree and outdegree.

Proof:-

Let D be eulerian and T be an eulerian trail in D. Each occurence (occurence at origin and terminus of T together is to be considered as a single occurence) of a given point v in T contributes one to $d^-(v)$ and one to $d^+(v)$.

Since each arc of D occurs exactly once in T, the contribution of each arc of D to $d^-(v)$ and $d^+(v)$ can be accounted in this way.

Hence $d^-(v) = d^+(v)$ for every point v of D.

Converse part:-

Conversely, let $d^-(v) = d^+(v)$ for every point v of D. Since the trivial digraph is vacously eulerian, let D have at least two points. Hence every point of D has positive indegree and out degree.

Hence D contains a cycle z. (since if you reach a point for the first time, you can always move out). The removal of the lines of z results in a spanning subdigraph D_1 in which again $d^-(v) = d^+(v)$ for every point of v. If D_1 has no arcs, then z is an eulerian trail in D. Otherwise, D_1 has a cycle z_1. Continuing the above process, when a digraph D_n with no arc is obtained, we have a partition of the arcs of D into n cycles, $n \geq 2$. Among these n cycles, takes two cycles z_i and z_j having a point v in common. The walk beginning at v and consisting of the cycles z_i and z_j in succession is a closed trail containing the lines of these two cycles. Continuing this process, we can construct a closed trail containing all the arcs of D.

Hence D is eulerian.

Paths and Connection:-

Definition:-

A walk (directed walk) in a digraph is a finite alternating sequence $W = v_0 x_1 v_1 \dots x_n v_n$ of vertices and arcs in which $x_i = (v_{i-1}, v_i)$ for every arc x_i. W is called a walk from v_0 to v_n or a $v_0 - v_n$ walk. The vertices v_0 and v_n are called origin and terminus of W respectively and $v_1, v_2 \dots v_{n-1}$ are called its internal vertices. The length of a walk is the number of occurrence of arcs in it. A walk in which the origin and terminus coincide is called a closed walk.

A path (directed path) is a walk in which all the vertices are distinct. A cycle (directed cycle or circuit) is a non trivial closed walk whose origin and internal vertices are distinct.

If there is a path from u to v then v is said to be reachable from u. A digraph is called strongly connected or disconnected or strong if every pair of points are mutually reachable.

 → strongly connec

A digraph is called unilaterally connected or unilateral if every pair of points at least one is reachable from the other. A digraph is called weakly connected or weak if the underlying graph is connected. A digraph is called disconnected if the underlying graph is disconnected.

The trivial digraph consisting just one point is (vacuously) strong since it does not contain two distinct points. Obviously strongly connected ⇒ unilaterally connected ⇒ weakly connected. But the converse is not true.

Definition:

Two digraphs $D_1 = (V_1, A_1)$ and $D_2 = (V_2, A_2)$ are said to be isomorphic (written $D_1 \cong D_2$) if there exists a bijection $f: V_1 \to V_2$ such that $(u, w) \in A_1$ iff $(f(u), f(w)) \in A_2$.

f is called an isomorphism from D_1 to D_2

Example:

The digraphs are isomorphic under the mapping f, where $f(1) = a$, $f(2) = b$, $f(3) = c$, $f(4) = d$.

Theorem: 10.2

If two digraphs are isomorphic then corresponding points have the same degree pair.

proof:

Let $D_1 = (V_1, A_1)$ and $D_2 = (V_2, A_2)$ be isomorphic under an isomorphism f. Let $v \in V_1$.

Let $N(v) = \{w / w \in V_1 \text{ and } (v, w) \in A_1\}$ and

$N\{f(v)\} = \{w / w \in V_2 \text{ and } (f(v), f(w)) \in A_2\}$.

Now, $w \in N(v) \iff (v, w) \in A_1$

$\iff \{f(v), f(w)\} \in A_2$

(since f is an isomorphism)

$\iff f(w) \in N(f(v))$

(by definition of $N(f(v))$)

Hence $|N(v)| = |N(f(v))|$ (since f is a bijection)

Here the L.H.S and R.H.S are respectively the outdegree of v and $f(v)$. Hence v and $f(v)$ have the same outdegree.

Similarly we can prove that v and $f(v)$ have the same indegree and hence v and $f(v)$ have the same degree pair.

Because of Theorem 10.1 and 10.2, it is obvious that two ~~as~~ isomorphic digraphs have the same number vertices and the same number of arcs.

Definition:

The converse diagraph D' of a digraph D is obtained from D by reversing the direction of each arc.

Obviously D and D' have same number of points and arcs. Moreover, the indegrees of a point v in D is equal to its outdegree in D' and vice versa.

Definition:

A digraph $D = (V, A)$ is called complete if for every pair of distinct points v and w in V, both (v, w) and (w, v) are in A.

This if a complete digraph has n vertices then it has $n(n-1)$ arcs.

Definition:

A digraph is called functional if every point has outdegree 1.

If a functional digraph has n vertices then the sum of the outdegrees of the points is n. Hence by the theorem: then 10.1

In a digraph D, sum of the indegrees of all the vertices is equal to the sum of their outdegrees, each sum being equal to the number of arc in D.

thm 10.1

Hence, the number of arcs in the digraph is n.